HANKEL TRANSFORMS

The object of this article is to introduce integral transform of a particular type, called the *Hankel transform,* and to illustrate the *use* of this method by means of examples. The treatment is that of a review article and as such is not meant to be exhaustive, its aim being to give a concatenated account of known results rather than present new ones. The emphasis throughout is on those results that are of frequent occurrence in boundary-value problems of mathematical physics, but some indication is also given for possible theoretical investigations.

Proofs are either omitted entirely or only the key steps are outlined. Readers interested in rigorous proofs of some of the statements in this article are referred to the books by Sneddon (1,2), Davies (3), Andrews and Shivamoggi (4), and Zayed (5).

The organization of the article is as follows: In the first section, we illustrate the motivation behind introducing the Hankel transform and then give a precise definition of the Hankel transform and its inversion. The next two sections are devoted to the derivation of some basic properties of Hankel transforms. In the following section, we explore the connection between Fourier and Hankel transforms. Parseval's relation for Hankel transforms is then deduced. We next introduce the modified operator of Hankel transforms. An overview of Erdelyi–Kober operators and their generalization by Sneddon and Cooke is given. We then derive Beltrami-type relations and give a brief account of their generalization by Sneddon. An extensive account is given of the applications of Erdelyi–Kober and Cooke operators to dual, triple, and quadruple integral equations involving Hankel transforms. A number of issues that arise in connection with applications of Hankel transforms to many physical problems is then addressed. For the convenience of the readers, a compendium is given in the last section of the basic theorems and formulas of Hankel transforms that are of frequent occurrence in applications.

THE HANKEL TRANSFORM

The Hankel transform arises naturally as a result of using the method of separation of variables to boundary value problems of mathematical physics in cylindrical coordinates, for example, boundary-value problems for the Laplace and Helmholtz equations involving half-spaces and regions bounded by parallel planes. In general, application of this technique is relevant to problems leading to the integration of equations of the type

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}-\frac{v^2}{r^2}\phi+L\phi=f(r,\ldots)$$

where L is a linear operator that does not contain r, and $f(r, \ldots)$ is a prescribed function.

To illustrate this, let us consider the axisymmetric solution $\phi(r, z)$ of Laplace's equation:

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}=0 \tag{1}$$

in the half-space $r > 0$, $z > 0$, which satisfies the boundary condition

$$\phi(r,0)=f(r) \tag{2}$$

where $f(r)$ is a prescribed function of r.

In addition, the solution of the problem must satisfy the regularity conditions so that the field decays as $R\to\infty$, where $R=\sqrt{r^2+z^2}$.

Assuming that the solution can be represented in the separated-variable form,

$$\phi(r,z)=\phi_1(r)\phi_2(z)$$

we find that Eq. (1) reduces to

$$\frac{1}{\phi_1}\frac{d^2\phi_1}{dr^2}+\frac{1}{\phi_1 r}\frac{d\phi_1}{dr}=\frac{-1}{\phi_2}\frac{d^2\phi_2}{dz^2} \tag{3}$$

Since the left-hand side of Eq. (3) depends only on r while the right-hand side only on z, we conclude that they must be equal to a constant, say, $\lambda = -s^2$, where s is a real quantity. Thus, we obtain two ordinary differential equations

$$\begin{aligned}\frac{d^2\phi_1}{dr^2}+\frac{1}{r}\frac{d\phi_1}{dr}+s^2\phi_1&=0\\ \frac{d\phi_2}{dz^2}-s^2\phi_2&=0\end{aligned} \tag{4}$$

The first of these equations is that of Bessel [see Watson (6)], whose solution bounded at the origin is

$$\phi_1(r)=A_1(s)J_0(sr)$$

where $A_1(s)$ is an arbitrary function of s and $J_0(sr)$ is the zeroth-order Bessel function of the first kind.

On the other hand, the solution of the second relation of Eq. (4) ensuring a decaying field is given by

$$\phi_2(z)=A_2(s)e^{-sz}$$

Therefore, the solution of Eq. (1) is

$$\phi(r,z)=A(s)J_0(sr)e^{-sz} \tag{5}$$

where $A(s)$ is an arbitrary function of s. Readers can easily verify that the other cases, viz., $\lambda = 0$ and $\lambda = s^2$ (s is a real quantity), must be ignored, since they do not ensure a decaying field as $R\to\infty$.

The solution of Eq. (5) has the property that, if $s > 0$, $\phi(r, z)\to 0$ as $R\to\infty$. By simple superposition, we can therefore construct the solution of the form

$$\phi(r,z)=\int_0^\infty sA(s)J_0(sr)e^{-sz}ds \tag{6}$$

The condition of Eq. (2) will be satisfied if

$$f(r) = \int_0^\infty sA(s)J_0(sr)\,ds \tag{7}$$

yielding an equation for determining the unknown function $A(s)$. It will be shown later that $A(s)$ is given by the formula

$$A(s) = \int_0^\infty rf(r)J_0(sr)\,dr \tag{8}$$

which upon substitution into Eq. (6) then formally gives the solution of our problem.

The formulas in Eqs. (7) and (8) define a transformation pair called the *Hankel transform of order zero.* We now give a formal definition of the Hankel transform of an arbitrary order of a function.

Given a real function $f(r)$ defined in the interval $(0, \infty)$, suppose that

1. $f(r)$ is piecewise continuous and of bounded variation in every finite subinterval $[a,b]$, where $0 < a < b < \infty$
2. the integral

$$\int_0^\infty \sqrt{r}|f(r)|\,dr < \infty$$

Then, *the Hankel transform of the νth order* of the function $f(r)$ satisfying the preceding conditions is defined as

$$\tilde{f}_\nu(s) = \int_0^\infty rf(r)J_\nu(sr)\,dr \tag{9}$$

which we shall write as

$$\tilde{f}_\nu(s) = \mathscr{H}_\nu[f(r); r \to s] \tag{10}$$

Sometimes, for the sake of brevity, we shall write this notation as $\mathscr{H}\nu[f(r);s]$, $\mathscr{H}\nu[f(r)]$, or simply $\mathscr{H}\nu f(r)$.

Readers should note that since the kernel of the Hankel transform is the Bessel function, the theory of Hankel transforms relies heavily on the theory of the Bessel functions. Perhaps, for this reason, in some literature, this transform is called Bessel transformation or Fourier–Bessel transformation.

The Hankel inversion theorem states that if the function $f(r)$ satisfies the preceding conditions, then

$$\int_0^\infty s\tilde{f}_\nu(s)J_\nu(sr)\,ds = f(r) \tag{11}$$

If the function has a jump discontinuity at a point, then the right-hand side of Eq. (11) should be replaced by the sum

$$\tfrac{1}{2}[f(r+0) + f(r-0)]$$

We shall not give a proof of the Hankel inversion theorem here. Interested readers are referred to the book by Sneddon (2).

It follows from Eqs. (10) and (11) that

$$f(r) = \int_0^\infty sJ_\nu(sr)\,ds \int_0^\infty r_0 f(r_0)J_\nu(sr_0)\,dr_0 \qquad 0 < r < \infty, \qquad \nu > -\tfrac{1}{2} \tag{12}$$

Equation (12) is called *Hankel's integral theorem.*

Evidently, Eq. (11) can be written as

$$f(r) = \mathscr{H}_\nu^{-1}[\tilde{f}_\nu(s); s \to r]$$

which, in the notation of Eq. (10), is equivalent to

$$f(r) = \mathscr{H}_\nu[\tilde{f}_\nu(s); s \to r]$$

whence establishing the rule $\mathscr{H}_\nu = \mathscr{H}_\nu^{-1}$. Thus, we see that if $\nu > -\frac{1}{2}$, there is a symmetrical relationship between a function and its Hankel transform of order ν, in the sense that if $\tilde{f}\nu(s)$ is the Hankel transform of order ν of a function $f(r)$, then $f(r)$is the Hankel transform of order ν of $\tilde{f}\nu(s)$.

Extensive tables have been constructed of the Hankel direct and inverse transforms of functions usually encountered in applications [for instance, see Erdelyi et al. (7)].

As in the case of other types of integral transforms, the use of Hankel transform has many advantages, for example, it is applicable to both homogeneous and inhomogeneous problems, it simplifies calculations and singles out the purely computational part of the solution, and it allows us to construct an operational calculus for a given kernel by using tables of direct and inverse transforms.

An extensive account of applications of the Hankel transform as well as other integral transforms to problems in mathematical physics was given by Sneddon (1,2,8) and Lebedev, Skalskaya, and Ufliand (9). Perhaps, it is Sneddon who may quite justifiably be regarded as the most ardent proponent of using the method of integral transforms—in particular, Hankel transform—to various boundary-value problems of mathematical physics.

SOME ELEMENTARY PROPERTIES OF HANKEL TRANSFORMS

Property 1

$$\mathscr{H}_{-m}[f(r); r \to s] = (-1)^m \mathscr{H}_m[f(r); r \to s] \qquad (m = \pm 1, \pm 2, \ldots, \pm n, \ldots)$$

Proof of this property follows from the fact that [Watson (6)]

$$J_{-m}(sr) = (-1)^m J_m(sr)$$

Property 2

$$\mathscr{H}_\nu[f(ar); r \to s] = a^{-2}\mathscr{H}_\nu\left[f(r); r \to \frac{s}{a}\right]$$

Proof. By definition, we have

$$\mathscr{H}_\nu[f(ar); r \to s] = \int_0^\infty rf(ar)J_\nu(sr)\,dr \tag{13}$$

By making a change of variable $ar = \rho$, we reduce the integral in Eq. (13) to the form

$$\begin{aligned}\mathscr{H}_\nu[f(ar); r \to s] &= a^{-2}\int_0^\infty \rho f(\rho) J_\nu(sa^{-1}\rho)\, d\rho \\ &= a^{-2}\mathscr{H}_\nu\left[f(r); r \to \frac{s}{a}\right]\end{aligned}$$

Property 3

$$\mathscr{H}_\nu[r^{-1}f(r); r \to s] = \frac{s}{2\nu}[\tilde{f}_{\nu-1}(s) + \tilde{f}_{\nu+1}(s)] \qquad (\nu \neq 0)$$

Proof. From the recurrence relation for the Bessel functions [Watson (6)]

$$j_{\nu-1}(x) - \frac{2\nu}{x}J_\nu(x) + J_{\nu+1}(x) = 0$$

we deduce

$$\begin{aligned}\mathscr{H}_\nu[r^{-1}f(r); r \to s] &= \int_0^\infty f(r) J_\nu(sr)\, dr \\ &= \frac{s}{2\nu}\left(\int_0^\infty rf(r)J_{\nu-1}(sr)\, dr \right.\\ &\quad \left. + \int_0^\infty rf(r)J_{\nu+1}(sr)\, dr\right) \\ &= \frac{s}{2\nu}[\tilde{f}_{\nu-1}(s) + \tilde{f}_{\nu+1}(s)]\end{aligned}$$

Property 4 The shift formula for the Hankel transforms is

$$\mathscr{H}_n[f(r-a)H(r-a); r \to s] = \sum_{m=-\infty}^{\infty} \alpha_m \tilde{f}_m(s)$$

where

$$\begin{aligned}\alpha_m &= J_{n-m}(sa) + \tfrac{1}{2}as[(m+1)^{-1}J_{n-m-1}(sa) \\ &\quad + (m-1)^{-1}J_{n-m+1}(sa)]\end{aligned}$$

Proof of this property is given in the book by Sneddon (2).

It should be mentioned here that it is not possible to obtain a simple shift formula for the Hankel transforms. This is primarily because the addition formula for the Bessel functions, that is, the Neumann–Lommel addition formula [Watson (6)]

$$J_n(x+y) = \sum_{m=-\infty}^{\infty} J_m(x)J_{n-m}(y)$$

is much more complicated than the addition formula for the exponential functions e^x and e^{ix} for the Laplace and Fourier transforms.

THE HANKEL TRANSFORMS OF DERIVATIVES OF A FUNCTION

In applications of Hankel transforms to physical problems, it is necessary to have expressions for the Hankel transforms of the derivatives of a function or a combination of them, through the Hankel transforms of the function itself. Using the definition of Hankel's transform and the formula for integrating by parts, we obtain

$$\begin{aligned}\mathscr{H}_\nu\left[\frac{df}{dr}; s\right] &= \int_0^\infty r\frac{df}{dr}J_\nu(sr)\, dr \\ &= [rf(r)J_\nu(sr)]_0^\infty - \int_0^\infty \frac{\partial}{\partial r}[rJ_\nu(sr)]f(r)\, dr\end{aligned} \qquad (14)$$

The first term on the right vanishes provided that the function $f(r)$ is such that

$$\lim_{r\to 0} r^{\nu+1}f(r) = 0, \qquad \lim_{r\to\infty} \sqrt{r}f(r) = 0$$

It follows from the arguments leading to the proof of the Hankel inversion theorem [see Sneddon (2)] that the second of these conditions holds for any $f(r)$ whose Hankel transform exists. Therefore, the first term on the right in Eq. (14) vanishes if

$$f(r) = o(r^{-\nu-1}), \qquad r \to 0$$

where o is the Landau's symbol of order.

From the theory of Bessel functions [Watson (6), Erdelyi et al. (10)], we have

$$\begin{aligned}\frac{\partial}{\partial r}[rJ_\nu(sr)] &= J_\nu(sr) + rJ'_\nu(sr) \\ J'_\nu(sr) &= srJ_{\nu-1}(sr) - \nu J_\nu(sr)\end{aligned}$$

so that Eq. (14) now takes the following form:

$$\mathscr{H}_\nu\left[\frac{df}{dr}; r\right] = (\nu-1)\int_0^\infty f(r)J_\nu(sr)\, dr - s\int_0^\infty rf(r)J_{\nu-1}(sr)\, dr \qquad (15)$$

However, the integral on the right is the $(\nu - 1)$th-order Hankel transform of $f(r)$, that is,

$$\int_0^\infty rf(r)J_{\nu-1}(sr)\, dr = \mathscr{H}_{\nu-1}[f(r); r \to s]$$

Thus, Eq. (15) takes the form

$$\mathscr{H}_\nu\left[\frac{df}{dr}; r \to s\right] = (\nu-1)\int_0^\infty f(r)J_\nu(sr)\, dr - s\mathscr{H}_{\nu-1}[f(r); r \to s] \qquad (16)$$

The first term on the right is obviously the νth-order Hankel transform of the function $r^{-1}f(r)$. However, our objective is to express everything in terms of the Hankel transform of the function $f(r)$. This can be achieved by utilizing the following relation [Erdelyi et al. (10)]:

$$J_\nu(sr) = \frac{1}{2\nu}[J_{\nu-1}(sr) + J_{\nu+1}(sr)] \qquad (17)$$

Inserting Eq. (17) into Eq. (16), after some arrangements, we finally obtain the following important relationship:

$$\begin{aligned}\mathscr{H}_\nu\left[\frac{df}{dr}; r \to s\right] &= -s\frac{\nu+1}{2\nu}\mathscr{H}_{\nu-1}[f(r); r \to s] \\ &\quad + s\frac{\nu-1}{2\nu}\mathscr{H}_{\nu+1}[f(r); r \to s]\end{aligned} \qquad (18)$$

Expressions for Hankel transforms of the higher derivatives of the function $f(r)$ may be deduced by repeated application of the formula in Eq. (18). For instance,

$$\mathscr{H}_\nu\left[\frac{d^2f}{dr^2}; r\to s\right] = \frac{s^2(\nu+1)}{4(\nu-1)}\mathscr{H}_{\nu-2}[f(r)] - \frac{s^2(\nu^2-3)}{2(\nu^2-1)}\mathscr{H}_\nu[f(r)] + \frac{s^2(\nu-1)}{4(\nu+1)}\mathscr{H}_{\nu+2}[f(r)] \tag{19}$$

In applications of Hankel transforms to many physical problems, it becomes necessary to have available the formula for Hankel transform of the differential operator:

$$\mathscr{B}_\nu = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{\nu^2}{r^2}$$

Integrating by parts and assuming that $df/dr = o(r^{-1})$, we find

$$\int_0^\infty r\frac{d^2f}{dr^2}J_\nu(sr)\,dr = -\int_0^\infty \frac{df}{dr}\frac{d}{dr}[rJ_\nu(sr)]\,dr$$

so that

$$\begin{aligned}\int_0^\infty r\left(\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr}\right)J_\nu(sr)\,dr &= -s\int_0^\infty \frac{df}{dr}rJ'_\nu(sr)\,dr\\ &= s\int_0^\infty f(r)\frac{d}{dr}[rJ'_\nu(sr)]\,dr\end{aligned} \tag{20}$$

Equation (20) was derived on the assumption that the function $rf(r) \to 0$ as $r \to 0$ or ∞.

We know from the theory of Bessel functions [Watson (6), Erdelyi et al. (10)] that the function $J_\nu(sr)$ satisfies the differential equation

$$\frac{d}{dr}[rJ'_\nu(sr)] = -\left(s^2 - \frac{\nu^2}{r^2}\right)rJ_\nu(sr) \tag{21}$$

Upon substitution of Eq. (21) into Eq. (20), we obtain the following formula:

$$\begin{aligned}\int_0^\infty r\left(\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{\nu^2}{r^2}f\right)J_\nu(sr)\,dr &= -s^2\\ \int_0^\infty rf(r)J_\nu(sr)\,dr &= -s^2\mathscr{H}_\nu[f(r); r\to s]\end{aligned} \tag{22}$$

An immediate consequence of Eq. (22) is the formula

$$\int_0^\infty r\left(\frac{d^2f}{dr^2} + \frac{1}{r}\frac{df}{dr}\right)J_0(sr)\,dr = -s^2\mathscr{H}_0[f(r); r\to s] \tag{23}$$

To illustrate the use of the properties of Hankel transforms, let us consider the classic problem of determining the potential at any point in the field induced by an electrified disk of radius a, whose potential is raised to ϕ_0 (ϕ_0 is a constant). The problem is known as *Weber's problem*. A discussion of this problem can be found in the books by Jeans (11) and Smythe (12). The problem reduces to that of solving Laplace's equation in Eq. (1) with the boundary conditions

$$\begin{aligned}\phi(r,0) &= \phi_0, \qquad 0 \le r < a\\ \left.\frac{\partial\phi}{\partial z}\right|_{z=0} &= 0, \qquad r > a\end{aligned} \tag{24}$$

The second boundary condition in Eq. (24) expresses the symmetry of the field with respect to the plane of the disk, that is, the plane $z = 0$.

To solve the problem, we use the zeroth-order Hankel transform of the function $\phi(r, z)$, that is,

$$\phi(r,z) = \mathscr{H}_0[\tilde{\phi}(s,z); s\to r] \tag{25}$$

Applying the transformation in Eq. (25) to Eq. (1) and making use of the relation of Eq. (23), we obtain the following ordinary differential equation

$$\frac{d^2\tilde{\phi}}{dz^2} - s^2\tilde{\phi} = 0$$

whose solution is

$$\tilde{\phi}(s,z) = A(s)e^{-sz} + B(s)e^{sz} \tag{26}$$

where $A(s)$ and $B(s)$ are some unknown functions of s.

Because of symmetry, it is sufficient to consider the half-space $z \ge 0$ only. Then, since the field must vanish at infinity (regularity conditions), we must set $B = 0$, so that Eq. (26) reduces to

$$\tilde{\phi}(s,z) = A(s)e^{-sz}$$

Therefore, our formal solution of the problem takes the form

$$\phi(r,z) = \mathscr{H}_0[A(s)e^{-sz}; s\to r] \tag{27}$$

Utilizing the boundary conditions in Eq. (24), we get the following equations to determine the unknown function $A(s)$:

$$\begin{aligned}\mathscr{H}_0[A(s); s\to r] &= \phi_0, \qquad 0 \le r < a\\ \mathscr{H}_0[sA(s); s\to r] &= 0, \qquad r > a\end{aligned}$$

or writing in integral form

$$\begin{aligned}\int_0^\infty sA(s)J_0(sr)\,ds &= \phi_0, \qquad 0 \le r < a\\ \int_0^\infty s^2A(s)J_0(sr)\,ds &= 0, \qquad r > a\end{aligned} \tag{28}$$

Equations of the type in Eq. (28) are called *dual integral equations*. A systematic treatment of this kind of equations will be discussed later. Here, we give a rather heuristic solution. Gradshteyn and Ryzhik (13) provide the following integrals:

$$\begin{aligned}\int_0^\infty \frac{\sin s}{s}J_0(sr)\,ds &= \frac{\pi}{2}, \qquad 0 \le r < a\\ \int_0^\infty (\sin s)J_0(sr)\,ds &= 0, \qquad r > a\end{aligned} \tag{29}$$

A comparison of Eqs. (28) with Eqs. (29) shows that the solution for $A(s)$ is

$$A(s) = \frac{2\phi_0}{\pi}\frac{\sin s}{s} \tag{30}$$

Putting Eq. (30) into Eq. (27), we obtain the solution of our problem as

$$\phi(r,z) = \frac{2\phi_0}{\pi}\int_0^\infty \frac{\sin s}{s} J_0(sr)e^{-sz}\,ds \tag{31}$$

The uniqueness of Eq. (31) follows from the physical contents of the problem.

RELATION BETWEEN FOURIER AND HANKEL TRANSFORMS

In this section, the relationship between Hankel and Fourier transforms of a function of two variables is explored. Specifically, we shall see that there exists a close relationship between the double Fourier transform of a function of two variables of a particular type and its Hankel transform.

Consider a function $f(x_1, x_2)$ that is a function of $r = x_1^2 + x_2^2$ only. The double Fourier transform $F(\alpha_1, \alpha_2)$ is

$$F(\alpha_1,\alpha_2) = \frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty f(\sqrt{x_1^2+x_2^2})e^{i(\alpha_1x_1+\alpha_2x_2)}\,dx_1\,dx_2 \tag{32}$$

If we make the substitutions into Eq. (32)

$$x_1 = r\cos\theta,\qquad x_2 = r\sin\theta,\qquad \alpha_1 = s\cos\varphi,\qquad \alpha_2 = s\sin\varphi$$

then, since

$$dx_1\,dx_2 = r\,dr\,d\theta,\quad \alpha_1x_1+\alpha_2x_2 = rs\cos(\theta-\varphi)$$

the double integral in Eq. (32) reduces to

$$F(\alpha_1,\alpha_2) = \frac{1}{2\pi}\int_0^\infty rf(r)\,dr\int_0^{2\pi} e^{irs\cos(\theta-\varphi)}\,d\theta \tag{33}$$

Since the inner integral on the right is 2π-periodic, it does not depend on φ, that is

$$\int_0^{2\pi} e^{irs\cos(\theta-\varphi)}\,d\theta = \int_0^{2\pi} e^{irs\cos\theta}\,d\theta$$

which is equal to $2\pi J_0(rs)$ [Watson (6), Erdelyi et al. (10)], where $s = \sqrt{\alpha_1^2 + \alpha_2^2}$. We therefore see that the function $F(\alpha_1, \alpha_2)$ is a function of s only and may be written as

$$F(s) = \int_0^\infty rf(r)J_0(sr)\,dr = \mathscr{H}_0[f(r); r\to s] \tag{34}$$

which, of course, is the zeroth-order Hankel transform of $f(r)$. On the other hand, by the Fourier inversion theorem, we have

$$f(x_1,x_2) = \frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty F(\alpha_1,\alpha_2)e^{-i(\alpha_1x_1+\alpha_2x_2)}\,d\alpha_1\,d\alpha_2$$

Using the same substitution as before, the preceding expression can be reduced to the following formula:

$$f(r) = \int_0^\infty sF(s)J_0(sr)\,ds = \mathscr{H}_0[F(s); s\to r] \tag{35}$$

Formulas in Eqs. (34) and (35) obviously express the Hankel inversion theorem in the special case where $\nu = 0$.

The preceding results can be easily generalized in case of n-dimensional Fourier transforms. If the function $f(x_1, x_2, \ldots, x_n)$ is a function only of $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$, then its Fourier transform $F(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a function of s only where $s = \sqrt{\alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2}$. More specifically, the following relationship holds:

$$s^{(n-1)/2}F(s) = \int_0^\infty r[r^{(n-1)/2}f(r)]J_{(n-1)/2}(sr)\,dr \tag{36}$$

For proof, the readers are referred to the book by Sneddon (1).

It therefore follows from Eq. (36) that $s^{(n-1)/2}F(s)$ is the Hankel transform of order $(n - 1)/2$ of the function $r^{(n-1)/2}f(r)$.

Similarly, by n-dimensional Fourier inversion theorem, it can be shown that

$$r^{(n-1)/2}f(r) = \int_0^\infty s[s^{(n-1)/2}F(s)]J_{(n-1)/2}(sr)\,ds \tag{37}$$

If we write

$$\phi(r) = r^{(n-1)/2}f(r),\qquad \tilde{\phi}_\nu(s) = s^{(n-1)/2}F(s),\qquad \nu = \frac{n-1}{2}$$

then Eqs. (36) and (37) take the following form:

$$\tilde{\phi}_\nu(s) = \int_0^\infty r\phi(r)J_\nu(sr)\,dr = \mathscr{H}_\nu[\phi(r); r\to s]$$
$$\phi(r) = \int_0^\infty s\tilde{\phi}_\nu(s)J_\nu(sr)\,ds = \mathscr{H}_\nu[\tilde{\phi}_\nu(s); s\to r]$$

The above formulas obviously define the νth-order Hankel transformation pair for the function $\phi(r)$.

PARSEVAL'S RELATION FOR HANKEL TRANSFORMS

Suppose that

$$\tilde{f}_\nu(s) = \mathscr{H}_\nu[f(r); r\to s],\qquad \tilde{g}_\nu(s) = \mathscr{H}_\nu[g(r); r\to s]$$

Then, putting formally, we obtain the equation

$$\begin{aligned}\int_0^\infty s\tilde{f}_\nu(s)\tilde{g}_\nu(s)\,ds &= \int_0^\infty s\tilde{f}_\nu(s)\,ds\int_0^\infty xg(x)J_\nu(sx)\,dx\\ &= \int_0^\infty xg(x)\,dx\int_0^\infty s\tilde{f}_\nu(s)J_\nu(sx)\,ds\end{aligned} \tag{38}$$

in which the inner integral, by Hankel's inversion theorem, is obviously equal to $f(r)$.

Equation (38) then yields the following formula:

$$\int_0^\infty s\tilde{f}_\nu(s)\tilde{g}_\nu(s)\,ds = \int_0^\infty xf(x)g(x)\,dx \tag{39}$$

The expression in Eq. (39) is evidently the Parseval relation for the Hankel transform. As in the case of other integral transforms, such as Fourier, Laplace, Mellin, and Kantorovich-Lebedev transforms, Parseval's relation is a very useful tool in many theoretical and practical investigations.

It should be noted here that a general Parseval relation involving Hankel transforms of two functions of different orders does not exist. This is primarily because the Neumann-Rahman formula (6,14) for the product of two first-kind Bessel functions of different orders,

$$J_{m+n}(sr)J_n(sr_0) = \frac{1}{\pi}\int_0^\pi \left\{\cos(n\varphi)T_m\left(\frac{r-r_0\cos\varphi}{R}\right) + \frac{r_0\sin n\varphi\sin\varphi}{R} \times U_{m-1}\left(\frac{r-r_0\cos\varphi}{R}\right)\right\} J_m(R)\,d\varphi, \qquad U_{-1}(\cdots)=0 \quad (40)$$

where $R = \sqrt{r^2 + r_0^2 - 2rr_0\cos\varphi}$, $T_m(\cdots)$ and $U_{m-1}(\cdots)$ are the Chebyshev polynomials of the first and second kinds, respectively, is much more complicated than the simplest rule for the product of two exponential functions (kernels of Laplace and Fourier transforms) of different powers.

As an example of application of Parseval's relation in Eq. (39), let us evaluate the integral

$$\mathscr{H}_\nu[x^{-2}J_\nu(ax); x\to s], \qquad \nu > -\tfrac{1}{2}$$

Taking $f(x) = x^\nu H(a - x)$ $(a > 0)$ and $g(x) = x^\nu H(b - x)$ $(b > 0)$, where $\mathscr{H}(\cdots)$ is the step function, we have

$$\tilde{f}_\nu(s) = \int_0^a x^{\nu+1}J_\nu(sx)\,dx; \qquad \tilde{g}_\nu(s) = \int_0^b x^{\nu+1}J_\nu(sx)\,dx$$

These integrals are easily evaluated [Gradshteyn and Ryzhik (13)] as

$$\tilde{f}_\nu(s) = \frac{a^{\nu+1}}{s}J_{\nu+1}(sa), \qquad \tilde{g}_\nu(s) = \frac{b^{\nu+1}}{s}J_{\nu+1}(sb)$$

Now, using Parseval's relation in Eq. (39), we obtain

$$(ab)^{\nu+1}\int_0^\infty s^{-1}J_{\nu+1}(sa)J_{\nu+1}(sb)\,ds = \int_0^{\min(a,b)} x^{2\nu+1}\,dx$$

Assuming that $0 < a < b$, we find that (13)

$$\int_0^\infty s^{-1}J_{\nu+1}(sa)J_{\nu+1}(sb)\,ds = \frac{1}{2(\nu+1)}\left(\frac{a}{b}\right)^{\nu+1} \qquad 0<a<b, \qquad \nu > -\tfrac{1}{2}$$

It therefore follows from the preceding equation that

$$\mathscr{H}_\nu[x^{-2}J_{\nu+1}(ax); x\to s] = \begin{cases} \dfrac{1}{2\nu}\left(\dfrac{s}{a}\right)^\nu, & 0<s<a \\ \dfrac{1}{2\nu}\left(\dfrac{a}{s}\right)^\nu, & s>a \end{cases}$$

where $\nu > \frac{1}{2}$.

THE HANKEL OPERATOR

In many theoretical investigations, it is more convenient to use a modified operator of Hankel transform, $S_{\eta,\alpha}$, instead of the operator $\mathscr{H}_\nu$. This modified Hankel operator is defined by the formula

$$S_{\eta,\alpha}[f(t);x] = 2^\alpha x^{-\alpha}\mathscr{H}_{2\eta+\alpha}[t^{-\alpha}f(t); t\to x] \qquad (41)$$

so that

$$S_{\eta,\alpha}[f(t);x] = 2^\alpha x^{-\alpha}\int_0^\infty t^{1-\alpha}f(t)J_{2\eta+\alpha}(xt)\,dt \qquad (42)$$

If we write

$$\tilde{f}_{\eta,\alpha}(x) = S_{\eta,\alpha}[f(t);x] \qquad (43)$$

then from Eq. (41), we obtain

$$\mathscr{H}_{2\eta+\alpha}[t^{-\alpha}f(t);x] = 2^{-\alpha}x^\alpha\tilde{f}_{\eta,\alpha}(x) \qquad (44)$$

Applying Hankel's inversion, we deduce from Eq. (43) that

$$f(t) = 2^{-\alpha}t^\alpha\mathscr{H}_{2\eta+\alpha}[x^\alpha\tilde{f}_{\eta,\alpha}(x);t]$$

or writing out the above expression in full, we obtain

$$f(t) = S_{\eta+\alpha,-\alpha}[\tilde{f}_{\eta,\alpha}(x);t]$$

thus establishing the rule

$$S_{\eta,\alpha}^{-1} = S_{\eta+\alpha,-\alpha} \qquad (45)$$

In applications, the following relationship is useful:

$$S_{\eta,\alpha}f(x) = 2^{-\lambda}x^\lambda S_{\eta\lambda/2,\alpha+\lambda}[x^\lambda f(x)]$$

the validity of which can be easily proved by writing out both sides of the equation using the definition in Eq. (42).

THE ERDELYI–KOBER OPERATORS OF FRACTIONAL INTEGRATION

In this section, we present a brief exposition of the so-called Erdelyi–Kober operators of fractional integrations (15–17) and their generalization due to Sneddon and Erdelyi (8,18) and Cooke (19,20). We next illustrate applications of these operators to the solution of dual, triple and quadruple integral equations involving Hankel transforms, that arise in many boundary value problems of mathematical physics, especially electrostatics and electromagnetic scattering. The description here closely follows Sneddon (21).

In a series of papers (15–17), Erdelyi and Kober investigated the properties of the fractional integral

$$\frac{x^{-\eta-\alpha+1}}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}t^{\eta-1}f(t)\,dt \qquad (\alpha>0, \qquad \eta>0)$$

which is a generalization of Riemann's integral

$$\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\,dt$$

and Weyl's integral

$$\frac{x^n}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}t^{-\alpha-\eta}f(t)\,dt \qquad (\alpha>0, \qquad \eta>0)$$

Definitions and Basic Results

If $\alpha > 0$, $\eta > -\frac{1}{2}$, we define the operator $I_{\eta,\alpha}$ by the equation

$$I_{\eta,\alpha}f(x) = \frac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)}\int_0^x (x^2-u^2)^{\alpha-1}u^{2\eta+1}f(u)\,du$$

$I_{\eta,0}$ is the identity operator, and if $\alpha < 0$, we define $I_{\eta,\alpha}$ by the relation

$$I_{\eta,\alpha}f(x) = x^{-2\eta-2\alpha-1}D_x^n x^{2\eta+2\alpha+2n+1}I_{\eta,\alpha+n}f(x)$$

where n is a positive integer such that $0 < \alpha + n < 1$ and D_x is the differential operator

$$D_x = \frac{d}{dx}x^{-1}$$

Similarly, if $\alpha > 0$, $\eta > -\frac{1}{2}$, we define the operator $K_{\eta,\alpha}$ by the equation

$$K_{\eta,\alpha}f(x) = \frac{2x^{2\eta}}{\Gamma(\alpha)}\int_x^\infty (u^2-x^2)^{\alpha-1}u^{-2\alpha-2\eta+1}f(u)\,du$$

$K_{\eta,0}$ is the identity operator, and if $\alpha < 0$, we define $K_{\eta,\alpha}$ by the equation

$$K_{\eta,\alpha}f(x) = (-1)^n x^{2\eta-1}D_x^n x^{2n-2+1}K_{\eta-n,\alpha+n}f(x)$$

Operators $I_{\eta,\alpha}$ and $K_{\eta,\alpha}$ are called Erdelyi–Kober operators.

We next establish some properties of these operators. If we assume that $\alpha > 0$, $\beta > 0$, we have

$$I_{\eta,\alpha}I_{\eta+\alpha,\beta}f(x) = \frac{2x^{-2\eta-2\alpha}}{\Gamma(\alpha)}\int_0^x (x^2-u^2)^{\alpha-1}u^{2\eta+1}du\,\frac{2u^{-2\eta-2\alpha-2\beta}}{\Gamma(\beta)} \times \int_0^u (u^2-t^2)^{B-1}t^{2\eta+2\alpha+1}f(t)\,dt$$

Interchanging the order of integration and using the result (13)

$$2\int_t^x (x^2-u^2)^{\alpha-1}(u^2-t^2)^{\beta-1}u^{-2\alpha-2\beta+1}\,du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}t^{-2\alpha}x^{-2\beta}(x^2-t^2)^{\alpha+\beta-1}$$

we obtain

$$I_{\eta,\alpha}I_{\eta+\alpha,\beta}f(x) = \frac{2x^{-2\eta-2\alpha-\beta}}{\Gamma(\alpha+\beta)}\int_0^x t^{2\eta+1}(x^2-t^2)^{\alpha+\beta-1}f(t)\,dt$$

The expression on the right is equal to $I_{\eta,\alpha+\beta}$, which follows from its definition, thus establishing the rule

$$I_{\eta,\alpha}I_{\eta+\alpha,\beta} = I_{\eta,\alpha+\beta} \tag{46}$$

Similarly, it can be shown that

$$K_{\eta,\alpha}K_{\eta+\alpha,\beta} = K_{\eta,\alpha+\beta} \tag{47}$$

The preceding relations are valid for $\alpha > 0$, $\beta > 0$, but it is a simple exercise to show that they are also valid for negative values of α and β. Also, it can be shown from the theory of integral equations of Abel type (8) that the inverse of the Erdelyi–Kober operators are given by the formulas:

$$I_{\eta,\alpha}^{-1} = I_{\eta+\alpha,-\alpha}, \qquad K_{\eta,\alpha}^{-1} = K_{\eta+\alpha,-\alpha} \tag{48}$$

The following formulas hold, whose validity can be proved very easily:

$$I_{\eta,\alpha}\{x^{2\beta}f(x)\} = x^{2\beta}I_{\eta+\beta,\alpha}f(x)$$
$$K_{\eta,\alpha}\{x^{2\beta}f(x)\} = x^{2\beta}K_{\eta+\beta,\alpha}f(x)$$

The following relationships hold between the Erdelyi-Kober and Hankel operators:

$$\begin{aligned} I_{\eta+\alpha,\beta}S_{\eta,\alpha} &= S_{\eta,\alpha+\beta}, & K_{\eta,\alpha}S_{\eta+\alpha,\beta} &= S_{\eta,\alpha+\beta} \\ S_{\eta+\alpha,\beta}S_{\eta,\alpha} &= I_{\eta,\alpha+\beta} & S_{\eta,\alpha}S_{\eta+\alpha,\beta} &= K_{\eta,\alpha+\beta} \\ S_{\eta+\alpha,\beta}I_{\eta,\alpha} &= S_{\eta,\alpha+\beta}, & S_{\eta,\alpha}K_{\eta+\alpha,\beta} &= S_{\eta,\alpha+\beta} \end{aligned} \tag{49}$$

The proofs of these identities are based on the properties of Bessel functions and are given in the book by Davies (3).

The Cooke Operators

Cooke (19,20) has defined the operators

$$\binom{b}{a}I_{\eta,\alpha}$$

and

$$\binom{d}{c}K_{\eta,\alpha}$$

by the formulas

$$\binom{b}{a}I_{\eta,\alpha}f(x) = \begin{cases} \dfrac{2x^{-2\alpha-2\eta}}{\Gamma(\alpha)}\displaystyle\int_a^b (x^2-u^2)^{\alpha-1}u^{2\eta+1}f(u)\,du, & \alpha>0 \\ f(x), & \alpha=0 \\ \dfrac{x^{-2\alpha-2\eta-1}}{\Gamma(1+\alpha)}\dfrac{d}{dx}\displaystyle\int_a^b (x^2-u^2)^{\alpha}u^{2\eta+1}f(u)\,du, & -1<\alpha<0 \end{cases} \tag{50}$$

for $0 < a < b < \infty$,

$$\binom{d}{c}K_{\eta,\alpha}f(x) = \begin{cases} \dfrac{2x^{2\eta}}{\Gamma(\alpha)}\displaystyle\int_c^d (u^2-x^2)^{\alpha-1}u^{-2\alpha-2\eta+1}f(u)\,du, & \alpha>0 \\ f(x), & \alpha=0 \\ \dfrac{-x^{2\eta-1}}{\Gamma(1+\alpha)}\dfrac{d}{dx}\displaystyle\int_c^d (u^2-x^2)^{\alpha}u^{-2\alpha-2\eta+1}f(u)\,du, & -1<\alpha<0 \end{cases} \tag{51}$$

for $0 < x < c < d$. It will be observed that these operators are related to the Erdelyi–Kober operators by the relations

$$\begin{pmatrix} x \\ 0 \end{pmatrix} I_{\eta,\alpha} = I_{\eta,\alpha}, \qquad \begin{pmatrix} \infty \\ x \end{pmatrix} K_{\eta,\alpha} = I_{\eta,\alpha}$$

Cooke (19,20) also defined the operators L and M by the equations

$$\begin{aligned} \begin{pmatrix} x, & b \\ c, & a \end{pmatrix} L_{\eta,\alpha} f(x) &= \begin{pmatrix} x \\ c \end{pmatrix} I_{\eta,\alpha}^{-1} \begin{pmatrix} b \\ a \end{pmatrix} I_{\eta,\alpha} f(x) \\ \begin{pmatrix} d, & b \\ x, & a \end{pmatrix} M_{\eta,\alpha} f(x) &= \begin{pmatrix} d \\ x \end{pmatrix} K_{\eta,\alpha}^{-1} \begin{pmatrix} b \\ a \end{pmatrix} K_{\eta,\alpha} f(x) \end{aligned} \tag{52}$$

and showed that if $a < b < c < x$,

$$\begin{aligned} \begin{pmatrix} x, & b \\ c, & a \end{pmatrix} L_{\eta,\alpha} f(x) = {} & \frac{2\sin(\pi\alpha)}{\pi} x^{-2\eta}(x^2 - c^2)^{-\alpha} \\ & \int_a^b \frac{(c^2 - t^2)^\alpha t^{2+1}}{x^2 - t^2} f(t)\, dt \end{aligned} \tag{53}$$

and that if $x < d < a < b$

$$\begin{aligned} \begin{pmatrix} d, & b \\ x, & a \end{pmatrix} M_{\eta,\alpha} f(x) = {} & \frac{2\sin(\pi\alpha)}{\pi} x^{2\eta+2\alpha}(d^2 - x^2)^{-\alpha} \\ & \int_a^b \frac{(t^2 - d^2)^\alpha t^{-2\alpha-2\eta+1}}{t^2 - x^2} f(t)\, dt \end{aligned} \tag{54}$$

BELTRAMI-TYPE RELATIONS

A classic problem of electrostatics concerns that of determining the potential of the electrostatic field due to a circular disk whose potential is prescribed. One way to solve this problem is to determine the charge density q on the disk and then to calculate the potential at any field point $\boldsymbol{r}$ by evaluating the integral

$$\int_S \frac{q(\boldsymbol{R}')}{|\boldsymbol{R} - \boldsymbol{R}'|} dS'$$

over the surface of the disk. In the case of axisymmetry, that is, when the prescribed potential $\phi(r)$ is a function of r only, Beltrami (22) showed that the density of the surface charge is given by the formula

$$q(r) = \frac{-1}{\pi r} \frac{d}{dr} \int_r^\infty \frac{x\, dx}{\sqrt{x^2 - r^2}} \frac{d}{dx} \int_0^x \frac{y\phi(y)\, dy}{\sqrt{x^2 - y^2}}, \qquad 0 \le r \le a \tag{55}$$

where a is the radius of the disk.

Sneddon (23) showed that Beltrami's relation in Eq. (55) is a special case of a general relation between Hankel transforms. In particular, he showed that the expression

$$\mathscr{H}_\mu[s^\delta \mathscr{H}_\nu f(s); r]$$

can be expressed as a double integral involving $f(r)$, which is a generalization of the integral occurring on the right hand side of Beltrami's relation in Eq. (55). By assigning particular values of the parameters μ, δ, and ν, we can deduce relations that are of interest in the investigations into axisymmetric boundary-value problems of potential theory.

If we apply the operator $K_{\eta-\gamma,\gamma}$ to both sides of the first equation of Eqs. (49) and make use of the second relation of Eqs. (49), we obtain

$$K_{\eta-\gamma,\gamma} I_{\eta+\alpha,\beta} S_{\eta,\alpha} = S_{\eta-\gamma,\alpha+\beta+\gamma} \tag{56}$$

Equation (56) can be written in terms of Hankel transforms as follows:

$$\begin{aligned} &\mathscr{H}_{2\eta+\alpha+\beta-\gamma}[t^{-\alpha-\beta-\gamma} f(t); r] \\ &\quad = \left(\frac{r}{2}\right)^{\alpha+\beta+\gamma} K_{\eta-\gamma,\gamma} I_{\eta+\alpha,\beta} 2^\alpha x^{-\alpha} \mathscr{H}_{2\eta+\alpha}[t^{-\alpha} f(t); x] \end{aligned} \tag{57}$$

For $\alpha = 0$, $\beta = (\mu - \nu - \delta)/2$, $\eta = \nu/2$, Eq. (57) simplifies significantly

$$\mathscr{H}_\mu[s^\delta \tilde{f}(s); r] = 2^\delta r^\delta K_{(\mu+\delta)/2,(\nu-\mu-\delta)/2} I_{\nu/2,(\mu-\nu-\delta)/2} f(r) \tag{58}$$

Some special cases of formulas in Eq. (58) are of particular interest. If we set $\mu = \nu$, we obtain

$$\mathscr{H}_\mu[s^\delta \tilde{f}(s); r] = 2^\delta r^{-\delta} K_{(\mu+\delta)/2,-\delta/2} I_{\nu/2,-\delta/2} f(r) \tag{59}$$

Special cases of particular interest are given by assigning $\delta = \pm 1$ to Eq. (59); we then obtain

$$\begin{aligned} \mathscr{H}_\nu[s\tilde{f}_\nu(s); r] &= \frac{-2}{\pi} r^{\nu-1} \frac{d}{dr} \int_r^\infty \frac{x^{1-2\nu}}{\sqrt{x^2 - r^2}} \frac{d}{dx} \int_0^x \frac{y^{\nu+1} f(y)\, dy}{\sqrt{x^2 - y^2}} \quad (\nu \ge 0) \\ \mathscr{H}_\nu[s^{-1}\tilde{f}(s); r] &= \frac{2}{\pi} r^\nu \int_r^\infty \frac{x^{-2\nu}}{\sqrt{x^2 - r^2}} \int_0^x \frac{y^{\nu+1} f(y)\, dy}{\sqrt{x^2 - y^2}} \quad (\nu \ge 0) \end{aligned} \tag{60}$$

On the other hand, if we put $\mu = \nu + 1$ in Eq. (59), we obtain the relation

$$\mathscr{H}_{\nu+1}[s^\delta \tilde{f}_\nu(s); r] = 2^\delta r^{-\delta} K_{(\nu+\delta+1)/2,(-1-\delta)/2} I_{\nu/2,(1-\delta)/2} f(r)$$

The special case $\delta = 1$ corresponds to the well-known formula

$$\mathscr{H}_{\nu+1}[s\tilde{f}_\nu(s); r] = -r^\nu \frac{d}{dr}[r^{-\nu} f(r)] \tag{61}$$

Expressions corresponding to the particular values 0 and -1 of δ are, respectively,

$$\begin{aligned} \mathscr{H}_{\nu+1}[\tilde{f}_\nu(s); r] &= \frac{-2}{\pi} r^\nu \frac{d}{dr} \int_r^\infty \frac{x^{-2\nu}\, dx}{\sqrt{x^2 - r^2}} \int_0^x \frac{y^{\nu+1} f(y)\, dy}{\sqrt{x^2 - y^2}} \quad (\nu \ge 0) \\ \mathscr{H}_{\nu+1}[s^{-1}\tilde{f}_\nu(s); r] &= r^{-\nu-1} \int_0^r u^{\nu+1} f(u)\, du \quad (\nu \ge 0) \end{aligned} \tag{62}$$

Finally, if we set $\mu = \nu - 1$ in Eq. (59), we obtain the relation

$$\mathscr{H}_{\nu-1}[s^\delta \tilde{f}_\nu(s); r] = 2^\delta r^{-\delta} K_{(\nu-1+\delta)/2,(\nu-1-\delta)/2} f(r) \tag{63}$$

The most frequently occurring special cases of the formula in Eq. (63) are

$$\mathcal{H}_{\nu-1}[s\tilde{f}_\nu(s); r] = r^{-\nu}\frac{d}{dr}[r^\nu f(r)] \qquad (\nu \geq 1)$$

$$\mathcal{H}_{\nu-1}[\tilde{f}_\nu(s); r] = \frac{2}{\pi} r^{\nu-1}\int_r^\infty \frac{x^{1-2\nu}\,dx}{\sqrt{x^2-r^2}}\frac{d}{dx}\int_0^x \frac{y^{\nu+1}f(y)\,dy}{\sqrt{x^2-y^2}} \qquad (\nu \geq 1)$$

$$\mathcal{H}_{\nu-1}[s^{-1}\tilde{f}_\nu(s); r] = r^{\nu-1}\int_r^\infty x^{1-\nu}f(x)\,dx \qquad (\nu \geq 1) \tag{64}$$

Beltrami's Relation for an Electrified Disk

As an application of Beltrami-type relations just derived, let us consider the problem of an electrified disk of radius a lying in the plane $z = 0$ with its center at the origin of the coordinate system. Let the surface charge density be $q(r)$. Then in the half-space $z \geq 0$ the potential of the electrostatic field will be $\phi_+(r, z)$ and in the half-space $z \leq 0$, it will be $\phi_-(r, z)$, where

$$\phi_\pm(r, z) = \mathcal{H}_0[\tilde{\phi}_0(s)e^{\pm sz}; r]$$

where

$$\tilde{\phi}_0(s) = \mathcal{H}_0[\phi(r, 0); s]$$

The charge density on the plane $z = 0$ is given by the equation

$$q(r) = \frac{-1}{4\pi}\left(\frac{\partial\phi_+}{\partial z} - \frac{\partial\phi}{\partial z}\right)_{z=0}$$

and it immediately follows from equation that

$$q(r) = \frac{1}{2\pi}\mathcal{H}_0[s\tilde{\phi}_0(s); r] \tag{65}$$

From the first equation of Eqs. (60) then we deduce Beltrami's relation in Eq. (55). On the other hand, we could write Eq. (65) in the form

$$\phi(r, 0) = 2\pi\mathcal{H}_0[s^{-1}\tilde{q}_0(s); r]$$

and then using the second relation of Eq. (60) deduce the equation

$$\phi(r, 0) = 4\int_r^\infty \frac{dx}{\sqrt{x^2-r^2}}\int_0^{\min(a,x)} \frac{yq(y)\,dy}{\sqrt{x^2-y^2}}$$

Interchanging the order of integration, the last equation can be written as

$$\phi(r, 0) = \int_0^a \sigma(y)K(r, y)\,dy$$

where

$$K(r, y) = 4y\int_{\min(r,y)}^\infty \frac{du}{\sqrt{(u^2-r^2)(u^2-y^2)}}$$

DUAL INTEGRAL EQUATIONS INVOLVING HANKEL TRANSFORMS

In the applications of the theory of Hankel transforms to the solution of boundary-value problems of mathematical physics, it often happens that the problem may be reduced to the solution of a pair of simultaneous equations of the form

$$f(x) = S_{\mu/2-\alpha,2\alpha}[1 + k(x)]\psi(x); \quad g(x) = S_{\nu/2-\beta,2\beta}\psi(x) \tag{66}$$

in which

$$f(x) = \begin{cases} f_1(x), & x \in I_1 = \{x: 0 < x < 1\} \\ f_2(x), & x \in I_2 = \{x: 1 < x < \infty\}\end{cases}$$

$$g(x) = \begin{cases} g_1(x), & x \in I_1 = \{x: 0 < x < 1\} \\ g_2(x), & x \in I_2 = \{x: 1 < x < \infty\}\end{cases}$$

The problem is as follows: Knowing the functions $k(x)$ [$k(x) \to 0$, $x \to \infty$], f_1, and g_2, is it possible to find the functions ψ, f_2, and g_1? In the following, we consider the special case where $k(x) = 0$, but it is straightforward to generalize the results for $k(x) \neq 0$.

To solve the problem, Sneddon proposed the following trial solution:

$$\psi(x) = S_{\nu/2+\beta,\mu/2-\nu/2-\alpha-\beta}h(x) \tag{67}$$

Putting Eq. (67) into Eqs. (65), we obtain

$$S_{\mu/2-\alpha,2\alpha}S_{\nu/2+\beta,\mu/2-\nu/2-\alpha-\beta}h = f$$
$$S_{\nu/2-\beta,2\beta}S_{\nu/2+\beta,\mu/2-\nu/2-\alpha-\beta}h = g$$

which can be rewritten, using the third and fourth relations of Eq. (49), as

$$I_{\nu/2+\beta,\nu/2-\nu/2+\alpha-\beta}h = f$$
$$K_{\nu/2-\beta,\mu/2-\nu/2-\alpha+\beta}h = g$$

whence

$$h = I^{-1}_{\nu/2+\beta,\mu/2-\nu/2+\alpha-\beta}f$$
$$h = K^{-1}_{\nu/2-\beta,\mu/2-\nu/2-\alpha+\beta}g \tag{68}$$

Writing Eqs. (68) on the intervals I_1 and I_2, we have

$$h_1(x) = \binom{x}{0} I^{-1}_{\nu/2+\beta,\mu/2-\nu/2+\alpha-\beta}f_1$$
$$h_2(x) = \binom{1}{0} I^{-1}_{\nu/2+\beta,\mu/2-\nu/2+\alpha-\beta}f_1 + \binom{x}{1} I^{-1}_{\nu/2+\beta,\mu/2-\nu/2+\alpha-\beta}f_2$$
$$h_2(x) = \binom{\infty}{x} K^{-1}_{\nu/2-\beta,\mu/2-\nu/2-\alpha+\beta}g_2 \tag{69}$$
$$h_1(x) = \binom{\infty}{1} K^{-1}_{\nu/2-\beta,\mu/2-\nu/2-\alpha+\beta}g_2 + \binom{1}{x} K^{-1}_{\nu/2-\beta,\mu/2-\nu/2-\alpha+\beta}g_1$$

Putting the first and third equations of Eqs. (69) into Eq. (67), we obtain the solution for $\psi(x)$. On the other hand, from the second and third equations of Eqs. (69), we deduce that

$$\binom{x}{1} I^{-1}_{\nu/2+\beta,\mu/2-\nu/2+\alpha-\beta}f_2 = \binom{\infty}{x} K^{-1}_{\nu/2-\beta,\mu/2-\nu/2-\alpha+\beta}g_2 - \binom{1}{0} I^{-1}_{\nu/2+\beta,\mu/2-\nu/2+\alpha-\beta}f_1$$

whence it follows by use of the L operator defined by Eq. (53) that

$$f_2 = \binom{x}{1} I_{\nu/2+\beta,\mu/2-\nu/2+\alpha-\beta} \binom{\infty}{x} K^{-1}_{\nu/2-\beta,\mu/2-\nu/2-\alpha+\beta} g_2 - \begin{pmatrix} x, & 1 \\ 1, & 0 \end{pmatrix} L_{\nu/2+\alpha,\nu/2-\mu/2+\beta-\alpha} f_1 \tag{70}$$

Thus, f_2 is determined. Similarly, using the first and fourth equations of Eqs. (68), we obtain for g_1 the formula:

$$g_1 = \binom{1}{x} K_{\nu/2-\beta,\mu/2-\nu/2-\alpha+\beta} I_{\nu/2+\beta,\mu/2-\nu/2+\alpha-\beta} f_1 - \begin{pmatrix} 1, & \infty \\ x, & 1 \end{pmatrix} M_{\nu/2-\beta,\mu/2-\nu/2-\alpha+\beta} g_2 \tag{71}$$

Thus, the first two equations in Eqs. (69) and Eqs. (70) and (71) give the complete solution to our problem. The same procedure, applied to the case where $k(x) \neq 0$, yields

$$\begin{aligned} h_1 + E(x) &= \binom{x}{0} I^{-1} f_1 \qquad (x \in I_1) \\ h_2 + E(x) &= \binom{1}{0} I^{-1} f_1 + \binom{x}{1} I^{-1} f_2 \qquad (x \in I_2) \\ h_2 &= \binom{\infty}{x} K^{-1} g_2 \qquad (x \in I_2) \\ h_1 &= \binom{\infty}{1} K^{-1} g_2 + \binom{1}{x} K^{-1} g_1 \qquad (x \in I_1) \end{aligned} \tag{72}$$

where

$$E(x) = S_{\mu/2-\alpha,\nu/2+\beta+\alpha-\mu/2} k S_{\nu/2+\beta,\mu/2-\nu/2-\alpha-\beta} h(x) \tag{73}$$

The subscripts with the I and K in Eqs. (72) are the same as those in Eqs. (69). Further details are carried out for the special case where $\nu = \mu$, $\beta = 0$, $g_2 = 0$, which is the most frequently occurring case in applications. In this case, we find from Eqs. (72) that $h_2(x) = 0$ and $h_1(x)$ solves the integral equation

$$h_1(x) + E(x) = \binom{x}{0} I^{-1}_{\nu/2,\alpha} f_1 \qquad (x \in I_1) \tag{74}$$

where

$$\begin{aligned} E(x) &\equiv S_{\nu/2-\alpha,\alpha} k S_{\nu/2,-\alpha} h(x) \\ &= 2^{\alpha} x^{-\alpha} \int_0^{\infty} t^{1-\alpha} k(t) 2^{-\alpha} t^{\alpha} J_{\nu-\alpha}(xt)\, dt \int_0^1 u^{1+\alpha} h_1(u) J_{\nu-\alpha}(tu)\, du \end{aligned}$$

and inverting the order of intergration, we have

$$E(x) = x^{-\alpha} \int_0^1 u^{1+\alpha} K(x,u) h_1(u)\, du \tag{75}$$

where

$$K(x,u) = \int_0^{\infty} t k(t) J_{\nu-\alpha}(xt) J_{\nu-\alpha}(ut)\, dt \tag{76}$$

Since the functions h_1 and h_2 have been determined, it is possible to find the functions ψ, f_2, and g_1 following the procedure for the case $k(x) = 0$. These details can be found in the papers by Sneddon (8) and Cooke (19,20).

An Example: Two Coaxial Electrified Circular Disks

The problem of two solid disks, each charged to a uniform potential ϕ_0, was the subject of numerous research starting with Love's paper (24) [for references see Cooke (19)]. If the disks have different potentials the problem may be reduced to two separate problems, in one of which the potentials are equal and in the other they are equal and opposite. Assume that the disks have the same radii, equal to unity, and are situated in the planes $z = 0$ and $z = h$, where r, θ, and z are cylindrical coordinates. Then, the problem reduces to that of solving Laplace's equation in Eq. (1) subject to the following boundary conditions:

$$\begin{aligned} &\phi(r,0) = \phi_0, \qquad 0 < r < 1 \\ &\phi(r,0_-) = \phi(r,0_+), \qquad 0 \le r < \infty \\ &\left.\frac{\partial\phi}{\partial z}\right|_{z=0-} = \left.\frac{\partial\phi}{\partial z}\right|_{z=0+}, \qquad r > 1 \\ &\phi(r,h) = \pm\phi_0, \qquad 0 < r < 1 \\ &\phi(r,h_-) = \phi(r,h_+), \qquad 0 \le r < \infty \\ &\left.\frac{\partial\phi}{\partial z}\right|_{z=h-} = \left.\frac{\partial\phi}{\partial z}\right|_{z=h+}, \qquad r > 1 \end{aligned} \tag{77}$$

The sign in fourth of the preceding conditions is positive or negative according to whether the disks are of like or unlike potentials ϕ_0.

The solution of the problem must satisfy the regularity conditions at infinity. Besides, in order to guarantee uniqueness of solution, it must satisfy the *edge condition* [see Meixner (25) and Mittra and Lee (26)] so that the electric energy stored in any neighborhood of the sharp edge $r = 1$ be finite, which imposes the restriction that the surface charge density not grow more rapidly than $\rho^{-1+\tau}$ with $\tau > 0$ as $\rho \to 0$, where $\rho = 1 - r$.

It can be shown by using zeroth-order Hankel transform to Laplace's equation in Eq. (1) that the electrostatic field can be represented by the potential function

$$\phi(r,z) = \mathsf{H}_0[\phi_0 s^{-1}(e^{-|z|s} + e^{-|z-h|s})A(s); s \to r] \tag{78}$$

which satisfies the second and fifth continuity conditions in Eqs. (77), the sign in Eq. (78) being positive or negative depending on whether the disks are of like or unlike potentials ϕ_0.

We find that the third and sixth conditions in Eqs. (77) will be satisfied if the function $A(s)$ satisfies the equation

$$\mathscr{H}_0[A(s); s \to r] = 0, \qquad r > 1 \tag{79}$$

Using the second and fourth boundary conditions in Eqs. (77) and Eqs. (79), we obtain the following dual integral equations:

$$\begin{aligned} \mathscr{H}_0[s^{-1}(1 \pm e^{-hs})A(s); s \to r] &= 1, \qquad 0 \le r < 1 \\ \mathscr{H}_0[A(s); s \to r] &= 0, \qquad r > 1 \end{aligned} \tag{80}$$

Using the modified operator of Hankel transform, we rewrite Eqs. (80) in the form

$$\begin{aligned} S_{-1/2,1}[1 \pm k(r)]A(r) &= 1, \qquad 0 \le r < 1 \\ S_{0,0}A(r) &= 0, \qquad r > 1 \end{aligned} \tag{81}$$

where $k(s) = e^{-hs}$. Thus, for our problem

$$\alpha = \tfrac{1}{2}, \qquad \mu = 0, \qquad \beta = 0, \qquad \nu = 0$$
$$f_1(r) = \frac{r}{2\phi_0}, \qquad g_2(r) = 0$$

Therefore, following the procedure outlined in the previous section, we find that $h_2(r) = 0$ and $h_1(r)$ solves the following integral equation:

$$h_1(r) + r^{-1/2}\int_0^1 u^{3/2}K(r,u)h_1(u)\,du = \binom{r}{0} I_{1/2,-1/2}f_1(r) \qquad 0 \le r < 1 \tag{82}$$

where

$$K(r,u) = \pm\int_0^\infty tk(t)J_{-1/2}(rt)J_{-1/2}(ut)\,dt$$

Writing $rh_1(r) = H(r)$, we reduce Eq. (82) to the following Fredholm integral equation of the second kind:

$$H(r) + \int_0^1 H(u)N(r,u)\,du = \binom{r}{0} I_{1/2,-1/2}f_1(r) \tag{83}$$

where

$$N(r,u) = \pm\sqrt{ru}\int_0^\infty tk(t)J_{-1/2}(rt)J_{-1/2}(ut)\,dt \tag{84}$$

The kernel $N(r, u)$ in Eq. (84) can be evaluated in closed form, namely,

$$N(r,u) = \pm\left(\frac{1}{(r+u)^2+h^2} + \frac{1}{(r-u)^2+h^2}\right) \tag{85}$$

The integral equation defined by Eqs. (83) and (85) can be solved numerically.

The surface density at any point of a disk in the plane $z = 0$ is equal to

$$\frac{-1}{4\pi}\left(\frac{\partial\phi}{\partial z}\right)_{z=0}$$

When both sides of the disk are taken into account, this gives for the total charge Q

$$\begin{aligned} Q &= \frac{\phi_0}{2\pi}\int_0^1 2\pi r\,dr\int_0^\infty A(s)J_0(sr)\,ds \\ &= \phi_0\int_0^1 rg_1(r)\,dr = \phi_0\int_0^1 r\binom{1}{r}K_{0,-1/2}h_1(r)\,dr \\ &= -\frac{\phi_0}{\sqrt{\pi}}\int_0^1 dr\,\frac{d}{dr}\int_x^1 \frac{u^2h_1(u)\,du}{\sqrt{u^2-1}} = \frac{2\phi_0}{\pi}\int_0^1 \frac{uH(u)\,du}{\sqrt{u^2-1}} \end{aligned} \tag{86}$$

Once the integral equation in Eq. (83) is solved for $H(r)$, the total charge can be found by evaluating the integral in Eq. (86) numerically and hence the capacity $C = Q/\phi_0$ can be found.

TRIPLE INTEGRAL EQUATIONS INVOLVING HANKEL TRANSFORMS

As an example of the use of Cooke operators, we consider the solution of certain triple integral equations involving Hankel transforms. The problem consists in finding a function $\Phi(\xi)$ satisfying

$$\begin{aligned} \int_0^\infty \Phi(\xi)J_\nu(\xi x)\,d\xi &= G_1(x), \qquad x \in I_1 \\ \int_0^\infty \xi^{-2\alpha}[1+k(\xi)]\Phi(\xi)J_\nu(\xi x)\,d\xi &= F_2(x), \qquad x \in I_2 \\ \int_0^\infty \Phi(\xi)J_\nu(\xi x)\,d\xi &= G_3(x), \qquad x \in I_3 \end{aligned} \tag{87}$$

where I_j (j = 1, 2, 3) denote, respectively, the intervals $(0, a)$, (a, b), and (b, ∞) with $0 < a < b$. The functions G_1, F_2, and G_3 are assumed to be prescribed. Assuming that

$$\Phi(\xi) = \xi\psi(\xi), \qquad f(x) = \left(\frac{2}{x}\right)^{2\alpha}F(x), \qquad g(x) = G(x)$$

and using the modified operator of the Hankel transform, we rewrite Eqs. (87) in the form

$$\begin{aligned} S_{\nu/2-\alpha,2\alpha}[\{1+k(\xi)\}\psi(\xi);x] &= f(x) \\ S_{\nu/2,0}\psi(\xi) &= g(x) \end{aligned} \tag{88}$$

We first consider the case where $k = 0$, $g_1 = g_3 = 0$, $|\alpha| < 1$. There are two different ways of solving Eqs. (88), one proposed by Sneddon and the other by Borodachev.

The Sneddon Trial Solution

Sneddon proposed the following solution for the equations in Eq. (88):

$$\psi = S_{\nu/2,-\alpha}h \tag{89}$$

Then, putting Eq. (89) into Eq. (88), we find that

$$\begin{aligned} S_{\nu/2,0}\psi &= K_{\nu/2,\alpha}h = g(x) \\ S_{\nu/2-\alpha,2\alpha}\psi &= I_{\nu/2,-\alpha}h = f(x) \end{aligned}$$

and solving for h we obtain

$$\begin{aligned} h &= I^{-1}_{\nu/2,-\alpha}f(x) \\ h &= K^{-1}_{\nu/2,\alpha}g(x) \end{aligned} \tag{90}$$

Now, suppose that $f(x) = f_1(x)$, $x \in I_1$, $f(x) = f_3(x)$, $x \in I_3$, and $g(x) = g_2(x)$, $x \in I_2$. We also write $h(x) = h_j(x)$, $x \in I_j$.

If we evaluate Eqs. (90) on I_3 and use $g_3 = 0$, we deduce that $h_3 = 0$. Similarly, if we evaluate Eq. (90) on I_1, we have

$$f_1(x) = \binom{x}{0} I_{\nu/2,\alpha}h_1(x) \tag{91}$$

and if we evaluate Eq. (90) on I_2 we have

$$h_2 = \binom{a}{0} I^{-1}_{\nu/2,\alpha} f_1 + \binom{x}{a} I^{-1}_{\nu/2,\alpha} f_2, \qquad x \in I_2 \tag{92}$$

Putting Eq. (92) into Eq. (91) and using the L operator defined by Eq. (53), we obtain

$$h_2 = -\begin{pmatrix} x, & a \\ a, & 0 \end{pmatrix} L_{\nu/2,\alpha} h_1 + \binom{x}{a} I^{-1}_{\nu/2,\alpha} f_2, \qquad x \in I_2 \tag{93}$$

Now, evaluating Eq. (90) on I_2 and I_1, respectively, we obtain the equations

$$g_2 = \binom{b}{x} K_{\nu/2,-\alpha} h_2, \qquad h_1 = \binom{b}{a} K^{-1}_{\nu/2,-\alpha} g_2 \tag{94}$$

Putting first of the relations in Eq. (94) into the second and using the M operator defined by Eq. (54), we obtain

$$h_1 = -\begin{pmatrix} a, & b \\ x, & a \end{pmatrix} M_{\nu/2,-\alpha} h_2 \qquad (x \in I_1) \tag{95}$$

Equations (93) and (95) form a pair of simultaneous equations for the unknown functions h_1 and h_2, but, by eliminating h_1 between them, we can derive a single Fredholm integral equation of the second kind for h_2. Solving it, we can determine h_1 using Eq. (95).

The same procedure applied formally to the case in which $k(\xi) \neq 0$, $g_1 = g_3 = 0$, $|\alpha| < 1$ leads to the set of simultaneous equations

$$\begin{aligned}
&h_1 + E = k_1 \qquad (x \in I_1) \\
&h_2 + E = -\begin{pmatrix} x, & a \\ a, & 0 \end{pmatrix} L_{\nu/2,\alpha} k_1 + \binom{x}{a} I^{-1}_{\nu/2,\alpha} f_2 \qquad (x \in I_2) \\
&h_1 = -\begin{pmatrix} a, & b \\ x, & a \end{pmatrix} M_{\nu/2,-\alpha} h_2 \qquad (x \in I_1)
\end{aligned} \tag{96}$$

where

$$\begin{aligned}
E(x) &\equiv S_{\nu/2-\alpha,\alpha} k S_{\nu/2,-\alpha} h \\
&= 2^{\alpha} x^{-\alpha} \int_0^\infty t^{1-\alpha} k(t) 2^{-\alpha} t^{\alpha} J_{\nu-\alpha}(xt)\, dt \\
&\quad \left(\int_0^a u^{1+\alpha} h_1(u) J_{\nu-\alpha}(tu)\, du + \int_a^b u^{1+\alpha} h_2(u) J_{\nu-\alpha}(tu)\, du \right. \\
&\quad \left. + \int_b^\infty u^{1+\alpha} h_3(u) J_{\nu-\alpha}(tu)\, du \right)
\end{aligned}$$

and inverting the order of integration in each of the three repeated integrals, we have

$$E(x) = x^{-\alpha} \int_0^\infty u^{1+\alpha} K(x,u) h(u)\, du \tag{97}$$

where

$$K(x,u) = \int_0^\infty t k(t) J_{\nu-\alpha}(xt) J_{\nu-\alpha}(ut)\, dt \tag{98}$$

Thus, we have three equations with three unknowns h_1, h_2, k_1. As before $h_3 = 0$. Solving for them, the unknown functions f_1 and f_3 can be found by the formulas

$$\begin{aligned}
f_1 &= \binom{x}{0} I_{\nu,\alpha} k_1 \\
f &= I_{\nu/2,\alpha} h + S_{\nu/2-\alpha,2\alpha} k S_{\nu/2,-\alpha} h
\end{aligned} \tag{99}$$

Equations (93) and (95) to (98) allow us to obtain the complete solution of the problem. [For further details readers are referred to the papers by Cooke (19,20,27,28)].

The Borodachev Trial Solution

Borodachev (31) developed a different trial solution to solve the triple integral equations (87). He argued as follows: Assume that the solution of the equations has the form

$$\psi(\xi) = S_{\beta,\gamma} h \tag{100}$$

Equations (88) for the case where $k(\xi) = 0$, may be reduced to the following form:

$$I_{\mu_1,\lambda_1} h = f, \qquad K_{\mu_2,\lambda_2} h = g$$

which occur when

$$S_{\nu/2-\alpha} S_{\beta,\gamma} = I_{\mu_1,\lambda_1}, \qquad S_{\nu/2,0} S_{\beta,\gamma} = K_{\mu_2,\lambda_2} \tag{101}$$

Using the third and fourth relations of Eqs. (49), we infer that

$$\begin{aligned}
&\beta + \gamma = \frac{\nu}{2} - \alpha, \qquad \mu_1 = \beta, \qquad \lambda_1 = 2\alpha + \gamma \\
&\beta = \frac{\nu}{2}, \qquad \mu_2 = \frac{\nu}{2}, \qquad \lambda_2 = \gamma
\end{aligned}$$

which yield

$$\begin{aligned}
&\beta = \frac{\nu}{2}, \qquad \gamma = -\alpha, \qquad \mu_1 = \frac{\nu}{2}, \qquad \mu_2 = \frac{\nu}{2}, \\
&\lambda_1 = \alpha, \qquad \lambda_2 = -\alpha
\end{aligned}$$

Thus, in this case Eq. (100) takes the form $\psi = S_{\nu/2,-\alpha} h$, that is, we have Sneddon's trial solution.

On the other hand, readers might note that Eqs. (88) can be reduced to the form

$$K_{\mu_3,\lambda_3} H = f, \qquad I_{\mu_4,\lambda_4} H = g \tag{102}$$

Carrying out calculations similar to the ones done, we have

$$\begin{aligned}
&\beta = \frac{\nu}{2} + \alpha, \qquad \gamma = -\alpha, \qquad \mu_3 = \frac{\nu}{2} - \alpha, \qquad \mu_4 = \frac{\nu}{2} + \alpha \\
&\lambda_3 = \alpha, \qquad \lambda_4 = -\alpha
\end{aligned} \tag{103}$$

Accordingly, in this case,

$$\psi = S_{\nu/2+\alpha,-\alpha} H \tag{104}$$

Equation (104) is called Borodachev's trial solution.

We will now use Borodachev's trial solution to reduce the triple integral equations in Eq. (88) to a Fredholm integral equation of the second kind. Substituting Borodachev's trial solution in Eq. (104) into Eqs. (88), we obtain [see Eqs. (102) and (103)]

$$K_{\nu/2-\alpha,\alpha}H = f, \qquad I_{\nu/2+\alpha,-\alpha}H = g$$

whence

$$H = K^{-1}_{\nu/2-\alpha,\alpha}f, \qquad H = I^{-1}_{\nu/2+\alpha,-\alpha}g \tag{105}$$

As before, for the sake of simplicity, we consider the case where $g_1 = g_3 = 0$. Then writing Eq. (105) for each interval, we obtain

$$\begin{aligned}
H_1 &= \binom{x}{0} I^{-1}_{\nu/2+\alpha,-\alpha}g_1 = 0\\
H_2 &= \binom{a}{0} I^{-1}_{\nu/2+\alpha,-\alpha}g_1 + \binom{x}{a} I^{-1}_{\nu/2+\alpha,-\alpha}g_2 = \binom{x}{a} I^{-1}_{\nu/2+\alpha,-\alpha}g_2\\
H_3 &= \binom{a}{0} I^{-1}_{\nu/2+\alpha,-\alpha}g_1 + \binom{b}{a} I^{-1}_{\nu/2+\alpha,-\alpha}g_2 + \binom{x}{b} I^{-1}_{\nu/2+\alpha,-\alpha}g_3\\
&= \binom{b}{a} I^{-1}_{\nu/2+\alpha,-\alpha}g_2\\
H_3 &= \binom{\infty}{x} K^{-1}_{\nu/2-\alpha,\alpha}f_3\\
H_2 &= \binom{\infty}{b} K^{-1}_{\nu/2-\alpha,\alpha}f_3 + \binom{b}{x} K^{-1}_{\nu/2-\alpha,\alpha}f_2
\end{aligned} \tag{106}$$

From the second and fourth formulas in Eqs. (106), we deduce that

$$g_2 = \binom{x}{a} I_{\nu/2+\alpha,-\alpha}H_2, \qquad f_3 = \binom{\infty}{x} K_{\nu/2-\alpha,\alpha}H_3 \tag{107}$$

Substituting Eq. (107) into the third and fifth equations in Eqs. (106) and making use of the operators L and M, we obtain the following system of equations:

$$\begin{aligned}
H_2 &= \binom{b}{x} K^{-1}_{\nu/2-\alpha,\alpha}f_2 - \begin{pmatrix} b, & \infty\\ x, & b\end{pmatrix} M_{\nu/2-\alpha,\alpha}H_3 \qquad (a < x < b)\\
H_3 &= -\begin{pmatrix} x, & b\\ b, & a\end{pmatrix} L_{\nu/2+\alpha,-\alpha}H_2 \qquad (b < x < \infty)
\end{aligned} \tag{108}$$

Using the definitions of the L and M operators, we see that the formulas in Eq. (108) constitute a pair of coupled integral equations, upon solving for which we can find the functions and H_2 and H_3, while $H_1 = 0$.

Putting the second formula of Eq. (108) into the first equation, we obtain a single integral equation of the second kind involving only H_2:

$$H_2(x) = \varphi(x) - \left(\frac{2}{\pi}\right)^2 \int_a^b K(x,y)H_2(y)\,dy \tag{109}$$

where

$$\varphi(x) = \binom{b}{x} K_{\nu/2,-\alpha}f_2$$

$$K(x,y) = \sin^2(\alpha\pi)\frac{x^\nu y^{1+2\alpha+\nu}}{(b^2-x^2)^\alpha(b^2-y^2)^\alpha}\int_0^\infty \frac{t^{1-2\nu-2\alpha}(t^2-b^2)^{2\alpha}}{(t^2-x^2)(t^2-y^2)}\,dt \qquad (-\tfrac{1}{2} < \alpha < 1) \tag{110}$$

An Example: An Electrified Annular Disk

To illustrate the application of Cooke's and Borodachev's solutions to the set of triple integral equations in Eq. (98), we consider the electrostatic field induced by an annular disk with internal and external radii a and b, respectively, the disk being charged to a potential equal to ϕ_0. The disk is assumed to lie in the plane $z = 0$.

The solution of the problem must satisfy Laplace's equation in Eq. (1) and the following boundary conditions:

$$\begin{aligned}
&\phi(r,0) = \phi_0, \qquad a < r < b\\
&\left.\frac{\partial\phi}{\partial z}\right|_{z=0} = 0, \quad 0 \le r < \alpha, \qquad b < r < \infty
\end{aligned} \tag{111}$$

Furthermore, the solution must satisfy the regularity condition and the edge conditions at the edges $r = a$ and $r = b$. As before, applying zeroth-order Hankel transform to the Eq. (1), it can be shown that the electrostatic potential is given by the equation

$$\phi(r,z) = \phi_0\mathscr{H}_0[s^{-1}A(s)e^{-sz}; s \to r] \tag{112}$$

where $A(s)$ is an unknown function of s to be determined. Equation (112) automatically satisfies the radiation conditions.

Making use of the boundary conditions in Eq. (111), we obtain the following triple integral equations:

$$S_{-1/2,1}A(r) = f(r), \qquad S_{0,0}A(r) = g(r)$$

where $f_2(r) = 2r/\phi_0$, $g_1(r) = 0$, $g_3(r) = 0$. Following Sneddon's trial solution in Eq. (89), we obtain the following Fredholm integral equation of the second kind:

$$\frac{x^2-\epsilon^2}{x^2}F(x) = 1 - \left(\frac{2}{\pi}\right)^2\int_\epsilon^1 K(x,y)F(y)\,dy \tag{113}$$

where

$$x = \frac{r}{a}, \qquad \epsilon = \frac{a}{b}, \qquad F(x) = h_2^*(xb)$$

$$K(x,y) = \frac{1}{2(x^2-y^2)}\left(\frac{x^2-\epsilon^2}{x}\log\frac{x+\epsilon}{x-\epsilon} - \frac{y^2-\epsilon^2}{y}\log\frac{y+\epsilon}{y-\epsilon}\right)$$

On the other hand, making use of Borodachev's trial solution in Eq. (104), we obtain the following Fredholm integral equation:

$$\frac{1-x^2}{x^2}G(x) = 1 - \left(\frac{2}{\pi}\right)^2\int_\epsilon^1 M(x,y)G(y)\,dy \tag{114}$$

where

$$G(x) = h_2^*(bx), \qquad h_2^*(r) = \frac{\sqrt{\pi} r^2}{2\phi_0\sqrt{b^2 - r^2}} h_2(r)$$
$$M(x, y) = \frac{1}{2(x^2 - y^2)} \left(\frac{1 - y^2}{y} \log \frac{1 + y}{1 - y} - \frac{1 - x^2}{x} \log \frac{1 + x}{1 - x} \right) \tag{115}$$

The surface charge density at any point of the disk is

$$q = \frac{-1}{4\pi}\left(\frac{\partial \phi}{\partial z}\right)_{z=0} = \frac{\phi_0}{4\pi} g_2(r)$$
$$= \begin{cases} \dfrac{1}{2\pi^2 r}\dfrac{d}{dr}\displaystyle\int_a^r \sqrt{\frac{b^2 - u^2}{r^2 - u^2}} h_2^*(u)\,du, & a < r < b \\ \dfrac{b\phi_0}{2\pi^2 r}\dfrac{d}{dr}\displaystyle\int_\epsilon^{r/b} \sqrt{\frac{1 - y^2}{r^2/b^2 - y^2}} G(y)\,dy, & a < r < b \end{cases} \tag{116}$$

Thus, the charge density at any point of the disk can be calculated once the integral equation in Eq. (114) is solved.

Considering both sides of the disk, the total charge is

$$Q = 4\pi \int_a^b rq(r, 0)\,dr = \frac{2\phi_0 b}{\pi\gamma}, \qquad \gamma^{-1} = \int_\epsilon^1 G(y)\,dy$$

whence

$$\phi_0 = \frac{\pi Q\gamma}{2b}$$

so that formula in Eq. (116) takes the form

$$q(r, 0) = \frac{\gamma Q}{2\pi r}\frac{d}{dr}\int_\epsilon^{r/b} \sqrt{\frac{1 - y^2}{r^2/b^2 - y^2}} G(y)\,dy, \qquad a < r < b \tag{117}$$

Of great interest is to find the asymptotic representation of the charge density $q(r, 0)$ as $r \to a + 0$ in the sense of Erdelyi, that is, the first term in the asymptotic expansion of $q(r, 0)$ as $r \to a + 0$. By letting $r \to a + 0$ in Eq. (117), we obtain

$$q(r, 0) \approx \frac{Q\omega_a(\epsilon)}{2\sqrt{2}\pi b^2}\left(\frac{r}{b} - \epsilon\right)^{-1/2}, \qquad r \to a + 0 \tag{118}$$

where

$$\omega_a(\epsilon) = \frac{\gamma}{\epsilon}\sqrt{\frac{1 - \epsilon^2}{\epsilon}} G(\epsilon) \tag{119}$$

Performing similar analyses on the Sneddon's trial solution, it can be shown that the surface charge density exhibits the following behavior as the outer contour of the disk is approached:

$$q(r, 0) = \frac{Q\omega_b(\epsilon)}{\sqrt{2}\pi b^2}\left(1 - \frac{r}{b}\right)^{-1/2}, \qquad r \to b - 0 \tag{120}$$

where

$$\omega_b(\epsilon) = \gamma\sqrt{1 - \epsilon^2} F(1) \tag{121}$$

Equations (118) to (121) show that the surface charge density exhibits a square-root singularity as the inner and outer edges of the disk are approached. Thus, edge conditions (Meixner's conditions) are satisfied.

Integral equations in Eqs. (113) and (114) admit closed-form solutions only in the special case where $\epsilon = 0$, that is, for the case of a circular disk:

$$F(x) = 1, \qquad G(x) = \frac{x}{\pi\sqrt{1 - x^2}} \log\frac{1 + x}{1 - x} \tag{122}$$

In the context of mathematically similar elastic contact problems, Borodachev (29) showed that the values of $G(x)$ do not differ practically in the range $0 \le \epsilon \le 0.5$. Therefore for this range, approximate values of the surface charge density can be calculated by using formulas in Eq. (122) while the integral equation in Eq. (113) can be solved to find the surface charge density for the range $0.5 < \epsilon < 1.0$.

Many other applications of the triple integral equations considered here to problems of electrostatics are given in Sneddon's book (8). It should be noted that using the same approach, it is possible to solve a wide variety of problems concerning diffraction of a plane electromagnetic wave by an annular disk and by a system of coaxial annular disks. Many examples of electromagnetic scattering by objects of different shapes are analyzed in the books by Bowman, Senior, and Uslenghi (30) and by Uslenghi (31).

QUADRUPLE INTEGRAL EQUATIONS INVOLVING HANKEL TRANSFORMS

We now use Cooke operators to reduce certain quadruple integral equations involving Hankel transforms to a Fredholm integral equation of the second kind or a system of those. The problem is to find a function $\psi(x)$ satisfying the equations

$$\begin{aligned} S_{\nu/2-\alpha,2\alpha}\psi(x) &= f_1(x), & x \in I_1 &= \{x{:}\,0 < x < \alpha\} \\ S_{\nu/2-\beta,2\beta}\psi(x) &= 0, & x \in I_2 &= \{x{:}\,a < x < b\} \\ S_{\nu/2-\alpha,2\alpha}\psi(x) &= f_3(x), & x \in I_3 &= \{x{:}\,b < x < c\} \\ S_{\nu/2-\beta,2\beta}\psi(x) &= 0, & x \in I_4 &= \{x{:}\,c < x < \infty\} \end{aligned} \tag{123}$$

Taking a trial solution in the form

$$\psi(x) = S_{\nu/2+\beta,-\alpha-\beta} h(x)$$

and then using the third and fourth relations from Eqs. (49), we obtain

$$\begin{aligned} f(x) &\equiv S_{\nu/2-\alpha,2\alpha}\psi(x) = I_{\nu/2+\beta,\alpha-\beta} h(x) \\ g(x) &\equiv S_{\nu/2-\beta,2\beta}\psi(x) = K_{\nu/2-\beta,\beta-\alpha} h(x) \end{aligned} \tag{124}$$

whence

$$\begin{aligned} h(x) &= I^{-1}_{\nu/2+\beta,\alpha-\beta} f(x) \\ h(x) &= K_{\nu/2-\beta,\alpha-\beta}\, g(x) \end{aligned} \tag{125}$$

Writing out Eqs. (125) on $I_j (j = 1, \ldots, 4)$, we have

$$h_1(x) = \binom{x}{0} I^{-1}_{\nu/2+\beta,\alpha-\beta} f_1 \qquad (x \in I_1)$$

$$h_2(x) = \binom{a}{0} I_{\nu/2+\beta,\alpha-\beta} f_1 + \binom{x}{a} I^{-1}_{\nu/2+\beta,\alpha-\beta} f_2 \qquad (x \in I_2)$$

$$h_3(x) = \binom{a}{0} I^{-1}_{\nu/2+\beta,\alpha-\beta} f_1 + \binom{b}{a} I^{-1}_{\nu/2+\beta,\alpha-\beta} f_2 + \binom{x}{b} I^{-1}_{\nu/2+\beta,\alpha-\beta} f_3 \qquad (x \in I_3)$$

$$h_4(x) = \binom{a}{0} I^{-1}_{\nu/2+\beta,\alpha-\beta} f_1 + \binom{b}{a} I^{-1}_{\nu/2+\beta,\alpha-\beta} f_2 + \binom{c}{b} I^{-1}_{\nu/2+\beta,\alpha-\beta} f_3 + \binom{\infty}{c} I^{-1}_{\nu/2+\beta,\alpha-\beta} f_4 \qquad (x \in I_4) \tag{126}$$

$$h_4(x) = \binom{\infty}{x} K^{-1}_{\nu/2-\beta,\alpha-\beta} g_4 = 0 \qquad (x \in I_4)$$

$$h_3(x) = \binom{c}{x} K^{-1}_{\nu/2-\beta,\alpha-\beta} g_3 \qquad (x \in I_3)$$

$$h_2(x) = \binom{c}{b} K^{-1}_{\nu/2-\beta,\alpha-\beta} g_3 \qquad (x \in I_2)$$

$$h_1(x) = \binom{c}{b} K^{-1}_{\nu/2-\beta,\alpha-\beta} g_3 + \binom{a}{x} K^{-1}_{\nu/2-\beta,\alpha-\beta} g_1 \qquad (x \in I_1)$$

From sixth equation of Eqs. (126), we have

$$g_3 = \binom{c}{b} K_{\nu/2-\beta,\beta-\alpha} h_3$$

which upon substitution into the seventh equation of Eq. (126) yields

$$h_2(x) = -\begin{pmatrix} b, & c \\ x, & b \end{pmatrix} M_{\nu/2-\beta,\beta-\alpha} h_3(x) \tag{127}$$

Writing Eq. (124) on I_3, we obtain

$$f_3(x) = \binom{a}{0} I_{\nu/2+\beta,\alpha-\beta} h_1 + \binom{b}{a} I_{\nu/2+\beta,\beta-\alpha} h_2 + \binom{x}{b} I_{\nu/2+\beta,\beta-\alpha} h_3 \tag{128}$$

Applying the operator

$$\binom{x}{b} I^{-1}_{\nu/2+\beta,\beta-\alpha}$$

to both sides of Eq. (128), we obtain

$$h_3(x) = \Lambda(x) + \begin{pmatrix} x, & b \\ b, & a \end{pmatrix} L_{\nu/2+\beta,\alpha-\beta} h_2(x) \tag{129}$$

where $\Lambda(x)$ is the known function given by

$$\Lambda(x) = \binom{x}{b} I^{-1}_{\nu/2+\beta,\alpha-\beta} f_3(x) - \begin{pmatrix} x, & a \\ b, & a \end{pmatrix} L_{\nu/2+\beta,\alpha-\beta} h_2(x) \tag{130}$$

Equations (127) and (129) constitute a pair of coupled integral equations for the determination of the unknown functions h_2 and h_3, but eliminating h_2, we obtain a single Fredholm equation of the second kind, namely,

$$h_3(x) + \mu \int_b^c K(x, x_0) h_3(x_0)\, dx_0 = \Lambda(x) \qquad (b < x < c) \tag{131}$$

where $\mu = (4/\pi^2) \sin^2[\pi(\alpha - \beta)]$ and the kernel is given by the equation

$$K(x, x_0) = x^{-\nu-2\beta}(x^2 - b^2)^{\beta-\alpha}(x_0^2 - b^2)^{\beta-\alpha} x_0^{2\alpha-\nu+1} \times \int_a^b \frac{(b^2 - y^2)^{2\alpha-2\beta} y^{2\nu-2\alpha+2\beta+1}}{(x_0^2 - y^2)(x^2 - y^2)}\, dy \qquad (b < x, x_0 < c) \tag{132}$$

Further details can be found in the article by Sneddon (21) and the references therein.

Quadruple integral equations of the type in Eq. (123) arise in many boundary-value problems of mathematical physics. For instance, the electrostatic problem of three coplanar circular disks charged to a uniform potential can be reduced to this kind of quadruple integral equations.

Finally, it should be noted that a new set of particular solutions can be derived for the quadruple integral equations of the type in Eq. (123) analogous to Borodachev's trial solution for the triple integral equations in Eq. (88), by assuming

$$\psi(x) = S_{\nu/2+\alpha,-\alpha-\beta} H(x) \tag{133}$$

Upon substituting Eq. (133) into Eqs. (123), we obtain

$$f(x) = K_{\nu/2-\alpha,\alpha-\beta} H(x)$$
$$g(x) = I_{\nu/2+\alpha,-\alpha+\beta} H(x)$$

MISCELLANEOUS

1. There is a generalization of the Hankel integral theorem in Eq. (12), known as Weber's integral [see Titchmarsh (32)]

$$f(r) = \int_0^\infty \frac{\varphi_s(r) s\, ds}{J_\nu^2(sa) + Y_\nu^2(sa)} \int_a^\infty r_0 f(r_0) \varphi_s(r_0)\, dr_0 \qquad a < r < \infty \tag{134}$$

involving the linear combination

$$\varphi_s(r) = J_\nu(sa) Y_\nu(sr) - Y_\nu(sa) J_\nu(sr) \tag{135}$$

of Bessel functions of the first and second kinds ($\nu > -\frac{1}{2}$). A sufficient condition for the validity of the Eq. (133) is that $f(r)$ be piecewise continuous and of bounded variation in every finite subinterval $[\alpha, \beta]$, where $a < \alpha < \beta < \infty$, and the integral

$$\int_a^\infty \sqrt{r} |f(r)|\, dr < \infty$$

It should be noted that Weber's integral reduces to Hankel's integral in the limit as $a \to 0$. Derivation of equations in Eqs. (134) and (135) is given in the famous book by Titchmarsh (32). Properties of Weber's transformation are also similar to those derived for Hankel transforms. Weber's transform is suited for solving equations of the form in Eq. (1) for domains with an excluded circular region. Below we illustrate the use of Weber's integral by one example.

Example. A cylindrical hole of radius a is drilled in an infinite body, and the walls of the hole are maintained at a temperature T_0 starting from the time $t = 0$. It is required to determine the temperature distribution in the body assuming that the initial temperature is zero.

The two-dimensional temperature distribution in the body is governed by the heat-conduction equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = \frac{\partial T}{\partial t}, \qquad a < r < \infty \tag{136}$$

satisfying the initial condition $T|_{t=0} = 0$ and the boundary and radiation conditions

$$T|_{r=a} = T_0, \qquad T|_{r\to\infty} \to 0$$

Multiplying both sides of Eq. (136) by $r\varphi_s(r)$ and integrating the resulting expression from a to ∞, we obtain

$$\frac{d\tilde{T}}{dt} - s^2\tilde{T} = \frac{2T_0}{\pi} \tag{137}$$

where

$$\tilde{T}(s,t) = \int_a^\infty r\tilde{T}(r,t)\varphi_s(r)\,dr$$

may be called the *Weber transform* of zeroth order of the function $T(r, t)$. In deriving Eq. (137), use has been made of the relations

$$\varphi_s(a) = 0, \qquad \varphi_s'(a) = \frac{2}{\pi a}$$

The solution of Eq. (137) satisfying the boundary condition $\tilde{T}|_{t=0}$ is

$$\tilde{T}(s,t) = \frac{2T_0}{\pi s^2}(1 - e^{-s^2 t})$$

Now, using Weber's inversion, we finally obtain the following formula for the temperature evolution in the body:

$$T(r,t) = \frac{2T_0}{\pi}\int_0^\infty \frac{(1-e^{-s^2t})\varphi_s(r)\,ds}{s[J_0^2(sa) + Y_0^2(sa)]} \tag{138}$$

Many other practical applications of Weber's integral are given in the book by Lebedev, Skalskaya, and Ufliand (9).

2. In applications of Hankel transforms to many physical problems, integrals of the following form are encountered:

$$\int_0^\infty s\tilde{f}_n(s)\tilde{F}_m(s)J_{m+n}(sr)\,ds \tag{139}$$

The need to evaluate such integrals arises in connection with the desire of transforming the solution for the physical quantities given in the space of Hankel transform domain into the physical space.

Using Parseval's relation in Eq. (39), we reduce the integral in Eq. (139) to the form

$$\int_0^\infty s\tilde{f}_n(s)\tilde{F}_m(s)J_{m+n}(sr)\,ds = \int_0^\infty r_0 f(r_0)\Phi(r_0)\,dr_0 \tag{140}$$

where

$$\begin{aligned}\Phi(r_0) &= \mathsf{H}_n[\tilde{F}_m(s)J_{m+n}(sr); s \to r_0]\\ &= \int_0^\infty s\tilde{F}_m(s)J_{m+n}(sr)J_n(sr_0)\,dr_0\end{aligned} \tag{141}$$

For the product of Bessel functions in Eq. (141), we use Neumann's formula generalized by Rahman (14) and then interchanging the order of integration, we get

$$\begin{aligned}\Phi(r_0) = \frac{1}{\pi}\int_0^\pi \Bigg[&\cos(n\phi)T_m\left(\frac{r - r_0\cos\phi}{R}\right)\\ &+ \frac{r_0 \sin n\phi \sin\phi}{R}U_{m-1}\left(\frac{r - r_0\cos\phi}{R}\right)\Bigg]F(R)\,d\phi\end{aligned} \tag{142}$$

In specific physical problems, however, the cases where $m = 0$ and $m = 1$, $n = 0$ are the most frequently encountered ones. In these cases, formula in Eq. (142) simplifies significantly. For instance, for $m = 0$, we have

$$\Phi(r_0) = \frac{1}{\pi}\int_0^\pi \cos(n\phi)F(R)\,d\phi$$

while for $m = 1$, $n = 0$, we have

$$\Phi(r_0) = \frac{1}{\pi}\int_0^\pi \frac{r - r_0\cos\phi}{R}F(R)\,d\phi$$

3. An efficient method of solving the integral equation (74) is based on representing the unknown function $h_1(x)$ in the form [Rahman (33)]

$$h_1(x) = x^{\nu-2\alpha}\sum_{n=0}^\infty a_n P_n^{\nu-\alpha,0}(1-2x^2) \tag{143}$$

where $P_n^{\nu-\alpha,0}(1 - 2x^2)$ is the Jacobi polynomial and a_n are the unknown expansion coefficients to be determined.

Putting the expansion in Eq. (143) into Eq. (73) and considering the orthogonality relationship for the Jacobi polynomials

$$\begin{aligned}&\int_0^1 \frac{P_n^{\alpha,\beta}(1-2x^2)P_m^{\alpha,\beta}(1-2x^2)\,dx}{2^{-2-\alpha-\beta}x^{-1-2\alpha}(1-x^2)^{-\beta}}\\ &\qquad = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!\Gamma(n+\alpha+\beta+1)(\alpha+\beta+2n+1)}\delta_{mn}\end{aligned}$$

(δ_{mn} – Kronecker's delta)

we obtain the following infinite system of linear algebraic equations:

$$\frac{a_m}{2(1+\nu-\alpha+2m)}+\sum_{n=0}^{\infty}a_nK_{mn}=r_m \qquad (m=0,1,2,\ldots,\infty) \qquad (144)$$

where

$$\begin{aligned}K_{mn}&=\int_0^{\infty}t^{-1}k(t)J_{1+\nu-\alpha+2m}(t)J_{1+\nu-\alpha+2n}(t)\,dt\\ r_m&=\int_0^1 x^{1+\alpha}r(x)P_m^{\nu-\alpha,0}(1-2x^2)\,dx\end{aligned} \qquad (145)$$

A key result that was used to obtain Eqs. (144) and (145) is the following integral [Rahman (34)]:

$$S_{a,-a}[(1-x^2)^{-b}P_n^{a,-b}(1-2x^2)\}=\frac{\Gamma(1-b+n)}{2^{a+b}n!x^{1-a-b}}J_{1+a-b+2n}(x)$$

The infinite system in Eq. (144) can be solved by truncation for the unknown expansion coefficients a_n.

In boundary-value problems, often it is often the case that the quantity $g_1(x)$ is of prime importance. For instance, in the charged disk problems, the function $g_1(x)$ is directly proportional to the surface charge density $q(r, 0)$ $(0 \le r < a)$, which, in turn, is essential for finding the capacitance. Rahman (33) showed that with the representation in Eq. (143), the function g_1 is given by

$$g_1(x)=x^{\nu}\sum_{n=0}^{\infty}a_n\frac{n!}{\Gamma(1-\alpha+n)}(1-x^2)^{-b}P_n^{\nu,-\alpha}(1-2x^2)$$

This method of solution is certainly preferrable to that based on using the numerical quadrature, because it bypasses the arduous job of evaluating Abel integrals numerically.

Furthermore, it was shown [Rahman (34)] that the following relation holds:

$$\begin{aligned}\begin{pmatrix}1\\x\end{pmatrix}K_{\nu/2,-\alpha}\left(\frac{x^{\nu-2\alpha}}{(1-x^2)^b}P_n^{\nu-\alpha,-b}(1-2x^2)\right)&=\frac{\Gamma(1-b+n)}{\Gamma(1-b-\alpha+n)}x^{\nu}(1-x^2)^{-b-\alpha}\\ &\quad\times P_n^{\nu,-b-\alpha}(1-2x^2)\end{aligned} \qquad (146)$$

Formula in Eq. (146) gives a class of *spectral relationship* for the operator $K_{\nu/2,-\alpha}$. It can be seen by writing out Eq. (146) in full that it gives a closed-form expression for a class of Abel integrals involving Jacobi polynomials. It can be used to a polynomial solution to Abel integral equations, which a number of boundary value problems of electrostatics can be reduced to.

4. The methods described in this article for solving dual integral equations are also applicable to a system of those of the form

$$\begin{aligned}S_{\mu_i/2-\alpha,2\alpha}\sum_{j=1}^{n}c_{ij}\psi_j(x)&=f_i(x), \qquad x\in I_1\\ S_{\nu_i/2-\beta,2\beta}\psi_i(x)&=g_i(x), \qquad x\in I_2\end{aligned}$$

   By a systematic use of the properties of Erdelyi–Kober operators Lowndes was able to show that the problem of solving a system of simultaneous equations of this type can be reduced to that of solving a system of simultaneous integral equations. Details of these results can be found in Sneddon's book (8). To the best of the writer's knowledge, generalization of these results has not yet been attempted for the case of simultaneous triple and quadruple integral equations.
5. The theory of Hankel transforms can also be extended to generalised functions or distributions via embedding theory or adjoint method. Interested readers are referred to consult the books by Zayed (5), Zemanian (34,35) and Brychkov and Prudnikov (36).

COMPENDIUM OF BASIC FORMULAS

For the sake of convenience of the readers, below we give a compendium of the basic formulas that are of frequent use in applications.

Definition of Hankel Transforms.

$$\tilde{f}_\nu(s)=\mathscr{H}_\nu[f(r),r\to s]=\int_0^{\infty}rf(r)J_\nu(sr)\,dr$$
$$f(r)=\mathscr{H}_\nu[\tilde{f}_\nu(s),s\to r]=\int_0^{\infty}s\tilde{f}_\nu(s)J_\nu(sr)\,dr$$

Some Properties of Hankel Transforms.

$$\mathscr{H}_{-m}[f(r),r\to s]=(-1)^m\mathscr{H}_m[f(r),r\to s] \qquad (m=\pm1,\pm2,\ldots,\pm n,\ldots)$$

$$\mathscr{H}_\nu[f(ar),r\to s]=a^{-2}\mathscr{H}_\nu\left[f(r),r\to\frac{s}{a}\right]$$

$$\mathscr{H}_\nu[r^{-1}f(r),r\to s]=\frac{s}{2\nu}[\tilde{f}_{\nu-1}(s)+\tilde{f}_{\nu+1}(s)] \qquad (\nu\neq0)$$

$$\mathscr{H}_n[f(r-a)H(r-a),r\to s]=\sum_{m=-\infty}^{\infty}\alpha_m\tilde{f}_m(s)$$

$$\alpha_m=J_{n-m}(sa)+\frac{1}{2}as[(m+1)^{-1}J_{n-m-1}(sa)+(m-1)^{-1}J_{n-m+1}(sa)]$$

$$\mathscr{H}_\nu[\mathscr{B}_\nu f(r),r\to s]=-s^2\mathscr{H}_\nu[f(r),r\to s]$$

$$\mathscr{B}_\nu=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{\nu^2}{r^2} \qquad (\nu=0,1,\ldots)$$

$$\mathscr{H}_\nu\left[r^{\nu-1}\frac{d}{dr}[r^{1-\nu}f(r)],r\to s\right]=-s\mathscr{H}_{\nu-1}[f(r),r\to s]$$

Parseval's Relation.

$$\int_0^{\infty}s\tilde{f}_\nu(s)\tilde{g}_\nu(s)\,ds=\int_0^{\infty}r_0f(r_0)g(r_0)\,dr_0$$

Modified Operator of Hankel Transform and Erdelyi-Kober Operators.

$$I_{\eta,0} = K_{\eta,0} = I$$
$$I_{\eta,\alpha}x^{2\beta}f(x) = x^{2\beta}I_{\eta+\beta,\alpha}f(x)$$
$$I_{\eta,\alpha}I_{\eta+\alpha,\beta} = I_{\eta,\alpha+\beta}$$
$$K_{\eta,\alpha}x^{2\beta}f(x) = x^{2\beta}K_{\eta-\beta,\alpha}f(x)$$
$$K_{\eta,\alpha}K_{\eta+\alpha,\beta} = K_{\eta,\alpha+\beta}$$
$$I_{\eta,-n}f(x) = x^{2n-2\eta-1}D_x^n x^{2\eta+1}f(x)$$
$$I_{\eta,\alpha}f(x) = x^{-2\eta-2n-1}D_x^n x^{2n+2\eta+2\alpha+1}I_{\eta,\alpha+n}f(x)$$
$$K_{\eta,-n}f(x) = (-1)^n x^{2\eta-1}D_x^n x^{2n-2\eta+1}f(x)$$
$$K_{\eta,\alpha}f(x) = (-1)^n x^{2\eta-1}D_x^n x^{2n-2\eta+1}f(x)K_{\eta-n,\alpha+n}f(x)$$
$$I_{\eta,\alpha}^{-1} = I_{\eta+\alpha,-\alpha}$$
$$K_{\eta,\alpha}^{-1} = K_{\eta+\alpha,-\alpha}$$
$$S_{\eta,\alpha}f(x) = 2^\alpha x^{-\alpha}H_{2\eta+\alpha}[t^{-\alpha}f(t), t \to x]$$
$$S_{\eta,\alpha}^{-1} = S_{\eta+\alpha,-\alpha}$$
$$S_{\eta,\alpha}f(x) = 2^{-\lambda}x^{\lambda}S_{\eta\lambda/2,\alpha+\lambda}[x^\lambda f(x)]$$
$$I_{\eta+\alpha,\beta}S_{\eta,\alpha} = S_{\eta,\alpha+\beta}$$
$$K_{\eta,\alpha}S_{\eta+\alpha,\beta} = S_{\eta,\alpha+\beta}$$
$$S_{\eta+\alpha,\beta}S_{\eta,\alpha} = I_{\eta,\alpha+\beta}$$
$$S_{\eta,\alpha}S_{\eta+\alpha,\beta} = K_{\eta,\alpha+\beta}$$
$$S_{\eta+\alpha,\beta}I_{\eta,\alpha} = S_{\eta,\alpha+\beta}$$
$$S_{\eta,\alpha}K_{\eta+\alpha,\beta} = S_{\eta,\alpha+\beta}$$

Some Beltrami-Type Relations of Common Occurence.

$$\mathscr{H}_0[s^{-1}\tilde{f}_0(s), s \to r] = \frac{2}{\pi}\int_r^\infty \frac{dt}{\sqrt{t^2-r^2}}\int_0^t \frac{xf(x)dx}{\sqrt{t^2-x^2}}$$
$$\mathscr{H}_0[s\tilde{f}_0(s), s \to r] = \frac{-2}{\pi r}\frac{d}{dr}\int_r^\infty \frac{t\,dt}{\sqrt{t^2-r^2}}\frac{d}{dt}\int_0^t \frac{xf(x)dx}{\sqrt{t^2-x^2}}$$

Some Useful Relations.

$$\int_0^\infty s\tilde{f}_n(s)\tilde{F}_m(s)J_{m+n}(sr)\,ds = \int_0^\infty r_0 f(r_0)\Phi(r_0)\,dr_0$$
$$\Phi(r_0) = \frac{1}{\pi}\int_0^\pi \left[\cos(n\phi)T_m\left(\frac{r-r_0\cos\phi}{R}\right) + \frac{r_0\sin n\phi\sin\phi}{R}U_{m-1}\left(\frac{r-r_0\cos\phi}{R}\right)\right]F(R)\,d\phi$$
$$J_{m+n}(sr)J_n(sr_0) = \frac{1}{\pi}\int_0^\pi \left[\cos(n\phi)T_m\left(\frac{r-r_0\cos\phi}{R}\right) + \frac{r_0\sin n\phi\sin\phi}{R}U_{m-1}\left(\frac{r-r_0\cos\phi}{R}\right)\right] \times J_m(sR)\,d\phi$$
$$S_{\eta,-\eta}[(1-x^2)^{-\alpha}P_n^{\eta,-\alpha}(1-2x^2)] = \frac{\Gamma(1-\alpha+n)}{2^{\eta+\alpha}n!x^{1-\eta-\alpha}}J_{1+\eta-\alpha+2n}(x)$$
$$\binom{1}{x}K_{\nu/2,-\alpha}\left(\frac{x^{\nu-2\alpha}}{(1-x^2)^\beta}P_n^{\nu-\alpha,-\beta}(1-2x^2)\right) = \frac{\Gamma(1-\beta+n)}{\Gamma(1-\beta-\alpha+n)}x^\nu(1-x^2)^{-\alpha-\beta}\times P_n^{\nu,-\alpha-\beta}(1-2x^2)$$

SUGGESTED FURTHER READING

Readers interested in rigorous proofs of various aspects of the theory of Hankel transforms are referred to the books by Sneddon (1,2), Davies (3), Andrews and Shivamoggi (4), and Zayed (5) and the papers by Erdelyi (15) and Erdelyi and Kober (16). Many applications of the theory of Hankel transforms to physical problems are given in the books by Sneddon (8) and Lebedev, Skalskaya, and Ufliand (9). Fractional integrals and derivatives and their applications to dual, triple, and quadruple integral equations involving Hankel transforms are discussed at greater length in Sneddon (8,21), Cooke (19,20,27,28), Borodachev (29) and Samko, Kilbas, and Marichev (37). Extension of the theory of Hankel transforms to generalized functions or distributions is presented in the books by Zayed (5), Zemanian (34,35), and Brychkov and Prudnikov (36).

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HARMONIC ANALYSIS. See FOURIER TRANSFORM.

HARMONIC DISTORTION MEASUREMENT. See ELECTRIC DISTORTION MEASUREMENT.

HARMONIC FACTOR. See POWER SYSTEM HARMONICS.