HARTLEY TRANSFORMS

Transform methods are used to determine the characteristics and to analyze the properties of a function describing a signal or a system that conveys information about or energy of a physical process. It is to be noted that transformation involves some sort of mathematical operation on the signal from one domain (time, space, or frequency) to another. Of all known transform methods, the most popular and widely used is the Fourier transform used in scientific and engineering applications. However, the Fourier transform is generally a complex function. Along with the Fourier transform, many researchers have proposed many Fourier-like transform methods such as the cosine transform, the sine transform, the Hilbert transform, and the Hartley transform, all of which provide some alternative methods of analyzing signals and can lead to an efficient implementation in some specific applications.

The key advantage of the Hartley transform is that it is real for any real signal. It works very much like the Fourier transform. In fact, there exists a very simple relationship between the Fourier transform and the Hartley transform. As a result, wherever the Fourier transform is being used, the Hartley transform can be used as well. Because the Hartley transform is a real function, in some cases, it offers considerable advantages over the Fourier transform. For this reason, the Hartley transform has attracted the attention of many researchers, who have found many applications for it in science and engineering.

For digital signal processing (DSP), the discrete version of the Hartley transform exists. It has spurred research on fast algorithms on the discrete Hartley transform, which are also called the fast Hartley transforms (FHT). To some extent, the discrete Hartley transform requires less time and memory or hardware compared to the discrete Fourier transform.

DEFINITIONS

In 1942, Hartley introduced a Fourier-like transform (later known as the Hartley transform), which can be described for a function $h(t)$ as

$$H(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t)\text{cas}(\omega t)\, dt \tag{1}$$

and the corresponding inverse transform can be described as

$$h(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(\omega)\text{cas}(\omega t)\, d\omega \tag{2}$$

where the integral kernel function cas(ωt) is defined as

$$\text{cas}(\omega t) = \cos(\omega t) + \sin(\omega t) \tag{3}$$

Some useful properties of the integral function, cas, are shown in Table 1. For the sake of convenience, a slightly different transform pair from the original Hartley transform pair is used in most literature and can be described for a function $h(t)$ as

$$H(f) = \int_{-\infty}^{\infty} h(t)\text{cas}(2\pi ft)\, dt \tag{4}$$

and

$$h(t) = \int_{-\infty}^{\infty} H(f)\text{cas}(2\pi ft)\, df \tag{5}$$

where the angular frequency ω and the radian frequency f are related by $\omega = 2\pi f$. Notice that the inverse integral function is the same as the direct integral function. It is evident from Eqs. (4) and (5) that the definition of the Hartley transform and the definition of the inverse Hartley transform are essentially the same. For our forthcoming discussions, we will refer to the second definition of the Hartley transform.

As is the case with the Fourier transform, the Hartley transform does not exist for all functions. In fact, the existence of the Hartley transform of a function is governed by the Dirichlet's conditions, which can be described for a function $h(t)$ as follows:

1. The function $h(t)$ must be absolutely integrable, that is

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty \tag{6}$$

   must hold.
2. The function $h(t)$ must have a finite number of maxima and minima and also must have a finite number of discontinuities in any interval.

Some useful signals, classified as power signals, do not satisfy the Dirichlet's conditions. However, their Hartley transforms can be expressed in terms of a special function called the Dirac delta function or the impulse function, which is used extensively for signal representation and analysis.

Table 1. Properties of the cas Function

1.	$\text{cas}(\alpha) = \cos(\alpha) + \sin(\alpha)$
2.	$\text{cas}(0) = 1$
3.	$\int \text{cas}(\alpha) = -\text{cas}(-\alpha)$
4.	$\frac{d}{d\alpha}\text{cas}(\alpha) = \text{cas}(-\alpha)$
5.	$\text{cas}^2(\alpha) + \text{cas}^2(-\alpha) = 2$
6.	$\cos(\alpha) = \frac{1}{2}[\text{cas}(\alpha) + \text{cas}(-\alpha)]$
7.	$\sin(\alpha) = \frac{1}{2}[\text{cas}(\alpha) - \text{cas}(-\alpha)]$
8.	$\text{cas}(\alpha_1 + \alpha_2) = \cos(\alpha_1)\,\text{cas}(\alpha_2) + \sin(\alpha_1)\,\text{cas}(-\alpha_2)$
9.	$\text{cas}(2\alpha) - \text{cas}(-2\alpha) = \text{cas}^2(\alpha) - \text{cas}^2(-\alpha)$
10.	$\text{cas}(2\alpha) = \text{cas}^2(\alpha) + \text{cas}(\alpha) - \text{cas}(-\alpha) - 1$
11.	$\text{cas}(\theta) = \frac{1-j}{2}e^{j\theta} + \frac{1+j}{2}e^{-j\theta}$
12.	$e^{j\theta} = \frac{1+j}{2}\text{cas}(\theta) + \frac{1-j}{2}\text{cas}(-\theta)$

HARTLEY TRANSFORMS OF ENERGY SIGNALS

Signals $h(t)$ for which $\int_{\infty}^{\infty} h^2(t)\, dt < \infty$ are classified as energy signals. Evidently, energy signals satisfy the Dirichlet's conditions. Here, we discuss the Hartley transform for some simple energy signals.

Rectangular Pulse

A rectangular pulse shown in Fig. 1, also called a gate function, is given by

$$\Pi(t/T) = \begin{cases} 1, & |t| < T/2 \\ 0, & \text{otherwise} \end{cases}$$

where T is the width of the pulse.

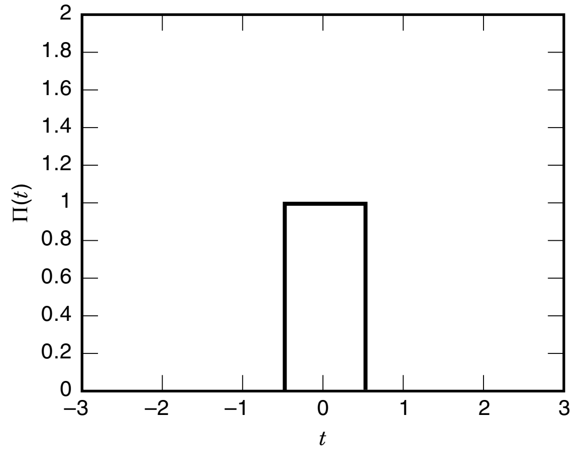

Figure 1. Rectangular pulse ($T = 1$).

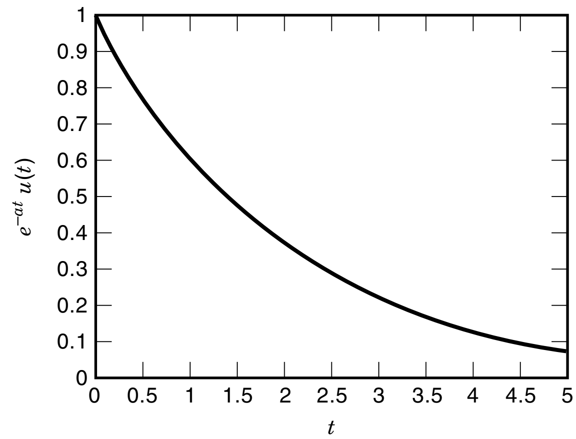

Figure 3. Exponential pulse ($a = 5$).

From the definition of the Hartley transform as stated in Eq. (4), the Hartley transform of $\Pi(t/T)$ is given by

$$\begin{aligned} G(f) &= \int_{-\infty}^{\infty} \Pi(t/T)\mathrm{cas}(2\pi ft)\,dt \\ &= \int_{-T/2}^{T/2} \mathrm{cas}(2\pi ft)\,dt \\ &= \frac{1}{2\pi f}\,[\sin(2\pi ft) - \cos(2\pi ft)]_{T/2}^{T/2} \\ &= \frac{\sin(\pi fT)}{\pi f} = T\mathrm{sinc}(fT) \end{aligned}$$

Note that the sinc function is defined as $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$. The plot of $G(f)$ is shown in Fig. 2. Notice that the Hartley transform of the gate function is the same as its Fourier transform. This is indeed true for all even functions. Notice from Fig. 2 that the first zero crossing of $G(f)$ occurs at frequency $f = 1/T$ and that as the pulse width T increases or decreases, the first zero crossing moves toward or away from the origin. In general, the shorter the duration of a signal, the wider its spectrum, and vice versa.

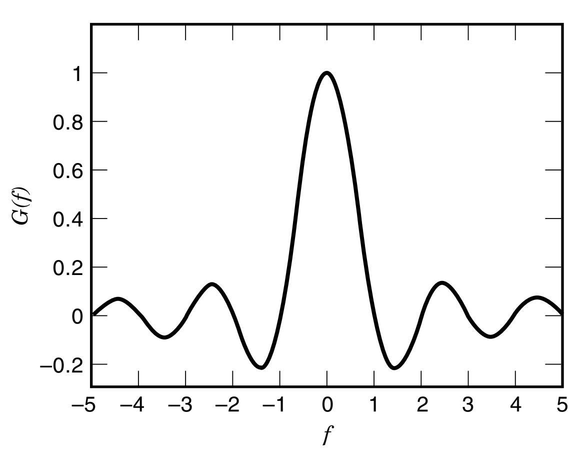

Figure 2. Hartley transform spectrum of a rectangular pulse ($T = 1$).

One-Sided Exponential Pulse

The analytic expression of a one-sided exponential pulse, shown in Fig. 3, is given by

$$h(t) = e^{-at}u(t)$$

where $u(t)$ is the unit step function and a is a constant that represents the rate of decay of the exponential pulse. The Hartley transform of $h(t)$ is $H(f)$ given by

$$\begin{aligned} H(f) &= \int_{-\infty}^{\infty} h(t)\mathrm{cas}(2\pi ft)\,dt \\ &= \int_{-\infty}^{\infty} e^{-at}u(t)\mathrm{cas}(2\pi ft)\,dt \\ &= \int_{0}^{\infty} e^{-at}\cos(2\pi ft\,dt + \int_{0}^{\infty} e^{-at}\sin(2\pi ft)\,dt \\ &= \frac{a + 2\pi f}{a^2 + 4\pi^2 f^2} \end{aligned}$$

The plot of $H(f)$ is shown in Fig. 4.

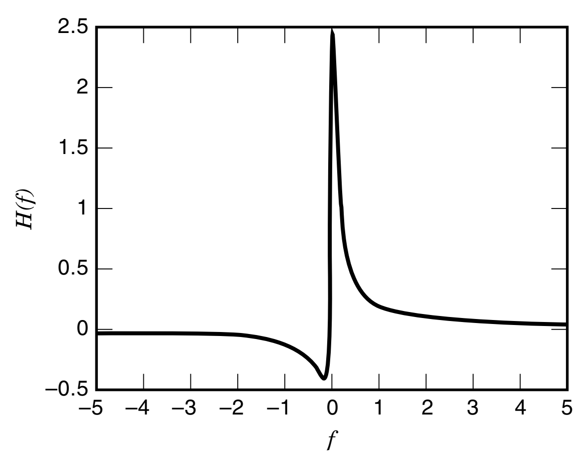

Figure 4. Hartley transform spectrum of an exponential pulse ($a = 5$).

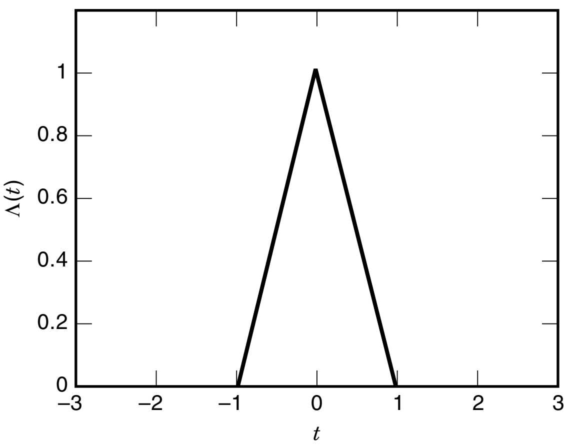

Figure 5. Triangular pulse ($T = 1$).

Triangular Pulse

The triangular pulse shown in Fig. 5 can be described as

$$\Lambda(t/T) = \begin{cases} 1 - |t|/T, & |t| < T \\ 0, & \text{otherwise} \end{cases}$$

where T represents the half of the width of the triangular pulse. The Hartley transform of the triangular pulse is

$$\begin{aligned} T(f) &= \int_{-\infty}^{\infty} \Lambda(t/T)\text{cas}(2\pi ft)\,dt \\ &= \int_{-T}^{0} (t/T+1)\text{cas}(2\pi ft)\,dt + \int_{0}^{T} (-t/T+1)\text{cas}(2\pi ft)\,dt \end{aligned}$$

After performing the integrations and simplifying, this expression yields the transform as

$$T(f) = T\text{sinc}^2(fT)$$

Figure 6 shows the plot of $T(f)$.

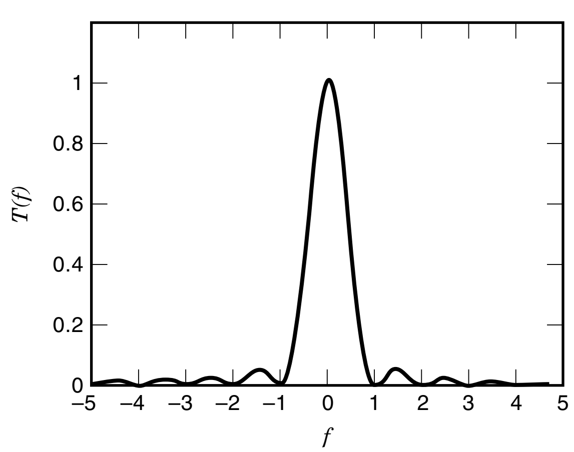

Figure 6. Hartley transform spectrum of a triangular pulse ($T = 1$).

Table 2. Hartley Transforms of Energy Signals

Function, $h(t)$	Hartley transform, $H(f)$
1. $e^{-at}u(t)$	$\dfrac{a + 2\pi f}{a^2 + 4\pi^2 f^2}$
2. $te^{-at}u(t)$	$\dfrac{a^2 + 4\pi af - 4\pi^2 f^2}{(a^2 + 4\pi^2 f^2)^2}$
3. $\prod(t/T) = \left(t + \dfrac{T}{2}\right) - u\left(t - \dfrac{T}{2}\right)$	$T\ \text{sinc}(fT)$
4. $\Lambda(t/T) = 1 - \dfrac{\vert t\vert}{T}, \vert t\vert < T$	$T\ \text{sinc}^2(fT)$
5. $e^{-a\vert t\vert}$	$\dfrac{2a}{a^2 + 4\pi^2 f^2}$
6. $\cos(2\pi f_0 t)\Pi(t/T)$	$\dfrac{T}{2}[\text{sinc}(T(f - f_0)) + \text{sinc}(T(f + f_0))]$
7. $e^{-a\vert t\vert}\cos(2\pi f_0 t)$	$\dfrac{a}{a^2 + 4\pi^2(f - f_0)^2} + \dfrac{a}{a^2 + 4\pi^2(f + f_0)^2}$
8. e^{-at^2}	$\sqrt{\dfrac{\pi}{a}}e^{\frac{-\pi^2 f^2}{a}}$
9. $\dfrac{1}{a^2 + t^2}$	$\dfrac{\pi}{a}e^{-2\pi a\vert f\vert}$

Table 2 lists the Hartley transforms of some frequently used energy signals including the rectangular pulse, the one-sided exponential pulse, and the triangular pulse. Hartley transforms of other energy signals can be obtained in the same way as discussed for the rectangular pulse, the one-sided exponential pulse, and the triangular pulse.

RELATIONSHIP BETWEEN THE HARTLEY AND THE FOURIER TRANSFORMS

Perhaps the most important property of the Hartley transform is its simple relationship with the Fourier transform. Note that the Fourier transform of a function $h(t)$ is defined by

$$F(f) = \int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}\,dt \tag{7}$$

and the inverse Fourier transform is defined by

$$h(t) = \int_{-\infty}^{\infty} F(f)e^{j2\pi ft}\,df \tag{8}$$

From the Euler's relation that $e^{j\theta} = \cos(\theta) + j\sin(\theta)$ and from the relations of the sine and cosine functions with the cas function as listed in Table 1, the Fourier transform $F(f)$ can be expressed as

$$\begin{aligned} F(f) &= \int_{-\infty}^{\infty} h(t)\cos(2\pi ft)\,dt - j\int_{-\infty}^{\infty} h(t)\sin(2\pi ft)\,dt \\ &= \int_{-\infty}^{\infty} h(t)\frac{\text{cas}(2\pi ft) + \text{cas}(-2\pi ft)}{2}\,dt \\ &\quad - j\int_{\infty}^{\infty} h(t)\frac{\text{cas}(2\pi ft) - \text{cas}(-2\pi ft)}{2}\,dt \\ &= \frac{H(f) + H(-f)}{2} - j\,\frac{H(f) - H(-f)}{2} \end{aligned} \tag{9}$$

where $H(f)$ is the Hartley transform of the function $h(t)$. It is obvious from Eq. (9) that if the Hartley transform of a function is known, the Fourier transform of the function can readily be obtained. In fact, there are some situations in which knowing the amplitude and the phase characteristics of a function separately is important. In those situations, the Fourier transform of the function can be obtained quickly from the Hartley transform in order to determine the amplitude and the phase spectrums of the function.

In the same way, using the Euler's relation, the Hartley transform $H(f)$ of the function $h(t)$ can be expressed as

$$\begin{aligned} H(f) &= \int_{-\infty}^{\infty} h(t)[\cos(2\pi ft) + \sin(2\pi ft)]\, dt \\ &= \int_{-\infty}^{\infty} h(t)\frac{e^{j2\pi ft} + e^{-j2\pi ft}}{2}\, dt + \int_{-\infty}^{\infty} h(t)\frac{e^{j2\pi ft} - e^{-j2\pi ft}}{2j}\, dt \\ &= \frac{1+j}{2}\int_{-\infty}^{\infty} h(t)e^{-j2\pi ft}\, dt + \frac{1-j}{2}\int_{-\infty}^{\infty} h(t)e^{j2\pi ft}\, dt \\ &= \frac{1+j}{2}F(f) + \frac{1-j}{2}F(-f) \end{aligned} \tag{10}$$

Thus, the Hartley transform of a function can readily be obtained by using the relation in Eq. (10) if the Fourier transform of the function is known. Fortunately, for many functions, the Fourier transforms are known.

PROPERTIES OF THE HARTLEY TRANSFORM

The Hartley transform provides an alternative representation of a function $h(t)$ from one domain to another. Obtaining the Hartley transform of the inverse Hartley transform from the definition is a straightforward task. Note that the information content in the Hartley transform of $h(t)$ and the information content in the function $h(t)$ itself are the same. But, one form or the other provides a better insight into the physical aspects of the signal or the system associated with it. Certain useful manipulations or operations such as scaling, shifting, and integration on the function cause distinctive changes in its corresponding Hartley transform, and vice versa. Some useful properties of the Hartley transform related to such operations are summarized in Table 3. These properties can be obtained from the definition of the Hartley transform and/or from the relationship between the Hartley transform and the Fourier transform. Proofs for some important properties follow.

Table 3. Properties of the Hartley Transform

1. Transformation	$h(t) \leftrightarrow H(f)$
2. Linearity	$a_1h_1(t) + a_2h_2(t) \leftrightarrow a_1H_1(f) + a_2H_2(f)$
3. Symmetry	$H(t) \leftrightarrow h(f)$
4. Scaling	$h(t/a) \leftrightarrow \lvert a\rvert H(af)$
5. Delay	$h(t - t_0) \leftrightarrow \cos(2\pi ft_0)H(f) + \sin(2\pi ft_0)H(-f)$
6. Modulation	$\cos(2\pi f_0t)h(t) \leftrightarrow \frac{1}{2}[H(f - f_0) + H(f + f_0)]$
7. Convolution	$h_1(t) \otimes h_2(t) \leftrightarrow \frac{1}{2}[H_1(f)H_2(f) + H_1(-f)H_2(f) + H_1(f)H_2(-f) - H_1(-f)H_2(-f)]$
8. Time differentiation	$\frac{d}{dt}h(t) \leftrightarrow -2\pi fH(-f)$
9. Time integration	$\int_{-\infty}^{t} h(\tau)\, d\tau \leftrightarrow \frac{H(-f)}{2\pi f} + \frac{H(0)\delta(f)}{2}$
10. Reversal	$h(-t) \leftrightarrow H(-f)$
11. Autocorrelation	$h(t) * h(t) \leftrightarrow \frac{1}{2}[H^2(f) + H^2(-f)]$
12. Multiplication	$h_1(t)h_2(t) \leftrightarrow \frac{1}{2}[H_1(f) \otimes H_2(f) + H_1(-f) \otimes H_2(f) + H_1(f) \otimes H_2(-f) - H_1(-f) \otimes H_2(-f)]$

Delay. If $h(t) \leftrightarrow H(f)$ represents the Hartley transform pair then

$$h(t - t_0) \leftrightarrow \cos(2\pi ft_0)H(f) + \sin(2\pi ft_0)H(-f)$$

Proof. From the definition, the Hartley transform of $h(t - t_0)$ is

$$h(t - t_0) \leftrightarrow \int_{-\infty}^{\infty} h(t - t_0)[\cos(2\pi ft) + \sin(2\pi ft)]\, dt$$

After substituting $t' = t - t_0$, $dt' = dt$, and $t = t_0 + t'$ and simplifying, we can obtain

$$\begin{aligned} h(t - t_0) &\leftrightarrow \int_{-\infty}^{\infty} h(t')[\cos(2\pi f(t' + t_0)) + \sin(2\pi f(t' + t_0))]\, dt' \\ &= \cos(2\pi ft_0)H(f) + \sin(2\pi ft_0)H(-f) \end{aligned}$$

Convolution. If $h_1(t) \leftrightarrow H_1(f)$ and $h_2(t) \leftrightarrow H_2(f)$ are two Hartley transform pairs, then the Hartley transform of the convolution of $h_1(t)$ and $h_2(t)$ is

$$\begin{aligned} h_1(t) \otimes h_2(t) &= \int_{-\infty}^{\infty} h_1(\tau)h_2(t - \tau)\, d\tau \\ &\leftrightarrow \frac{1}{2}[H_1(f)H_2(f) + H_1(-f)H_2(f) \\ &\quad + H_1(f)H_2(-f) - H_1(-f)H_2(-f)] \end{aligned}$$

Proof. From the definition

$$\begin{aligned} h_1(t) \otimes h_2(t) &\leftrightarrow \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h_1(\tau)h_2(t - \tau)\, d\tau\right]\mathrm{cas}(2\pi ft)\, dt \\ &= \int_{-\infty}^{\infty} h_1(\tau)\left[\int_{-\infty}^{\infty} h_2(t - \tau)\mathrm{cas}(2\pi ft)\, dt\right] d\tau \end{aligned}$$

Using the delay property and the relations between the cosine and the sine functions with the cas function as listed in Table 1,

$$\begin{aligned} &h_1(t) \otimes h_2(t) \\ &\quad\leftrightarrow \int_{-\infty}^{\infty} h_1(\tau)[\cos(2\pi f\tau)H_2(f) + \sin(2\pi f\tau)H_2(-f)]\, d\tau \\ &\quad= H_2(f)\int_{-\infty}^{\infty} h_1(\tau)\frac{\mathrm{cas}(2\pi f\tau) + \mathrm{cas}(-2\pi f\tau)}{2}\, d\tau \\ &\quad\quad + H_2(-f)\int_{-\infty}^{\infty} h_1(\tau)\frac{\mathrm{cas}(2\pi f\tau) - \mathrm{cas}(-2\pi f\tau)}{2}\, d\tau \\ &\quad= \frac{1}{2}[H_1(f)H_2(f) + H_1(-f)H_2(f) + H_1(f)H_2(-f) \\ &\quad\quad - H_1(-f)H_2(-f)] \end{aligned}$$

For some special cases, a simplified transform expression for the convolution of $h_1(t)$ and $h_2(t)$ can be obtained.

Case 1: If $h_1(t)$ or $h_2(t)$ is even or both are even, then

$$h_1(t) \otimes h_2(t) \leftrightarrow H_1(f)H_2(f)$$

Case 2. If $h_1(t)$ is odd, then

$$h_1(t) \otimes h_2(t) \leftrightarrow H_1(f)H_2(-f)$$

Case 3. If $h_2(t)$ is odd, then

$$h_1(t) \otimes h_2(t) \leftrightarrow H_1(-f)H_2(f)$$

Case 4. If both $h_1(t)$ and $h_2(t)$ are odd, then

$$h_1(t) \otimes h_2(t) \leftrightarrow -H_1(f)H_2(f)$$

Note that a function $h(t)$ is even if $h(t) = h(-t)$ or odd if $h(t) = -h(-t)$. Thus for the above-mentioned cases, the Hartley transform of the convolution of two functions can be obtained by a single multiplication from their transforms.

Power Spectrum. The power spectrum $P(f)$ of a function $h(t)$ is

$$P(f) = \frac{1}{2}[H^2(f) + H^2(-f)]$$

where $H(f)$ is the Hartley transform of $h(t)$.

Proof. If $F(f)$ is the fourier transform of $h(t)$, then

$$P(f) = F(f)F^*(f)$$

where $F^*(f)$ is the complex conjugate of $F(f)$. Hence, using relation in Eq. (9) between the Hartley and the Fourier transforms and simplifying, we can easily obtain

$$P(f) = \frac{1}{2}[H^2(f) + H^2(-f)]$$

Thus, finding the power spectrum of a signal from the Hartley transform is considerably easier than finding it from its Fourier transform becuse the process involves only real arithmetic and only two multiplications.

HARTLEY TRANSFORM OF POWER SIGNALS

So far we have considered only energy signals for the Hartley transform. These energy signals possess finite energy over the interval $(-\infty, \infty)$. Therefore, they are abolutely integrable and so satisfy the Dirichlet's conditions for the existence of $H(f)$. However, there is a class of signals, called power signals that are very useful but are not absolutely integrable. More rigorously, a power signal $f(t)$ has infinite energy but finite power such as the sine wave or the unit step function. This means that $f(t)$ does not satisfy the condition, $\int_{-\infty}^{\infty} f^2(t)\,dt < \infty$, but

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} f^2(t)\,dt < \infty$$

holds. It is possible to obtain the Hartley transform of these power signals if we allow impulse functions as part of the Hartley transform.

Impulse Function

The Hartley transform of the impulse function $\delta(t)$ is

$$H(f) = \int_{-\infty}^{\infty} \delta(t)\text{cas}(2\pi ft)\,dt = 1$$

Thus, we have the transform pair $\delta(t) \leftrightarrow 1$. Because of the symmetry in the Hartley and inverse Hartley transforms, we also have

$$1 \leftrightarrow \delta(f)$$

Thus the Hartley transform of unity is an impulse at the origin. Figure 7 shows the constant function and the corresponding Hartley transform.

The Signum Function

The signum function is defined as

$$\text{sgn}(t) = \begin{cases} -1, & t < 0 \\ 0, & t = 0 \\ 1, & t > 0 \end{cases}$$

Notice that if $h(t) \leftrightarrow H(f)$, then from the time differentiation property (see Table 3),

$$h^{(1)}(t) \leftrightarrow -2\pi f H(-f)$$

By differentiating the signum function, we obtain

$$\frac{d}{dt}\text{sgn}(t) = 2\delta(t)$$

If $H(f)$ denotes the Hartley transform of sgn(t), then from the differentiation property listed in Table 3, we obtain

$$(-2\pi f)H(-f) = 2$$

which yields

$$H(-f) = \frac{1}{-\pi f}$$

Hence by replacing $-f$ by f, we obtain

$$H(f) = \frac{1}{\pi f}$$

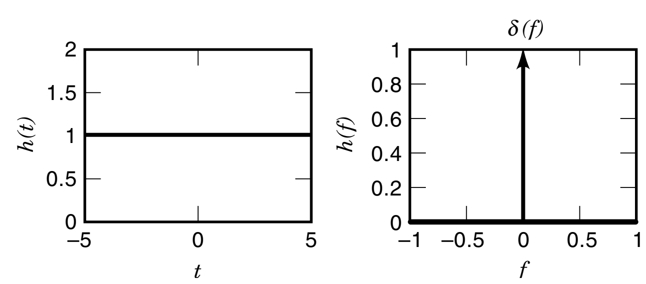

Figure 7. Constant function and its Hartley transform.

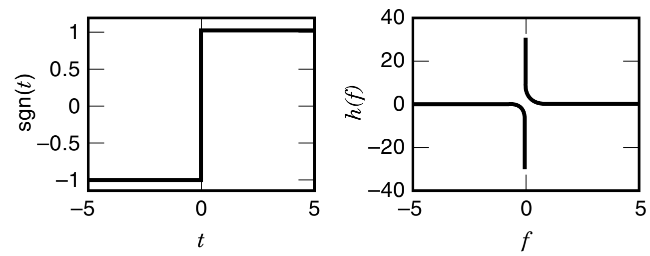

Figure 8. Signum function and its Hartley transform.

Figure 8 shows the plots of the signum function and its Hartley transform.

The Unit Step Function

The unit step function can be expressed in terms of the signum function as

$$u(t) = \frac{1}{2} + \frac{1}{2}\,\text{sgn}(t)$$

Thus, the transform pair for $u(t)$ is

$$u(t) \leftrightarrow \frac{1}{2}\,\delta(f) + \frac{1}{2\pi f}$$

The Hartley transform of the useful power signals are summarized in Table 4 and can be derived independently as demonstrated previously or from their known Fourier transforms.

The foregoing discussions on the Hartley transform were based on the integral definition or the continuous-time definition of the Hartley transform. The integral definition allows us to study many analytical properties as well as to develop theory and explore properties for the discrete version of the Hartley transform. The discrete Hartley transform has found popularity in many real-time DSP applications.

DISCRETE HARTLEY TRANSFORM

The discrete version of the Hartley transform (DHT) of a data sequence $x(n)$, $n = 0, 1, 2, \ldots, N-1$, is described as

$$H(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} h(n)\text{cas}(2\pi nk/N) \qquad (11)$$

Table 4. Hartley Transforms of Power Signals

$h(t)$	$H(f)$
1. $\delta(t)$	1
2. 1	$\delta(f)$
3. $u(t)$	$\frac{1}{2}\delta(f) + \frac{1}{2\pi f}$
4. $\text{sgn}(t)$	$\frac{1}{\pi f}$
5. $\cos w_0 t$	$\frac{1}{2}[\delta(f-f_0) + \delta(f+f_0)]$
6. $\sin w_0 t$	$\frac{1}{2}[\delta(f-f_0) - \delta(f+f_0)]$
7. $\sum_{k=-\infty}^{\infty} \delta(t-kT)$	$\frac{1}{2}\sum_{k=-\infty}^{\infty} \delta(f-k/T)$

for $k = 0, 1, 2, \ldots, N-1$. The corresponding inverse discrete Hartley transform is described as

$$h(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} H(k)\text{cas}(2\pi nk/N) \qquad (12)$$

The direct implementation of the discrete Hartley transform is computationally intensive when N is very large. Many fast Hartley transform algorithms can be found in Ref. 3 and are mostly based on the assumption that the number of data samples N is a power of 2. Most of the algorithms require either that data samples in the input sequence be sorted in bit-reversed order before processing or that the items in the transform sequence need to be sorted in bit-reversed order after processing.

The term bit-reversed ordering refers to finding the new index of an item by reversing the bits of the binary representation of the index of an element in given sequence and then placing the item in the new index position of the resulting sequence. For example, notice that three binary digits are required to index the data samples of a sequence $h(n) = \{h(0), h(1), h(2), h(3), h(4), h(5), h(6), h(7)\}$ where $N = 8$. If (n_2, n_1, n_0) represents the index of an item in $h(n)$, then the item should be copied in index position (n_0, n_1, n_2) in the bit-reversed sequence. Thus, after bit-reversed ordering, the resulting sequence will contain items in the order $h(0)$, $h(4)$, $h(2)$, $h(6)$, $h(1)$, $h(5)$, $h(3)$, $h(7)$.

Based on the Hou's fast Hartley transform algorithm described in Ref. 8, and algorithm for the fast computation of the Hartley transform is described here for the case when the transform size N is a power of two:

1. Perform permutation of data samples in sequence $x(n)$, $n = 0, 1, \ldots, N-1$ so that samples are in bit-reversed order.
2. Perform the following for $i = 2, 4, 8, 16, \ldots, N$ and for $j = 0, 2i, 3i, \ldots, N-i$:
   a. Copy $x(j), x(j+1), \ldots, x(j+i-1)$ to $g(0), g(1), \ldots, g(i-1)$, respectively.
   b. Perform the following for $k = 0, 1, \ldots, i/2 - 1$:

$$y(k) = g(k) + g(k+i/2)\cos(2\pi k/i) + g(i-k)\sin(2\pi k/i)$$

$$y(k+i/2) = g(k) - g(k+i/2)\cos(2\pi k/i) - g(i-k)\sin(2\pi k/i)$$

   c. Copy $y(0), y(1), \ldots, y(i-1)$ to $x(j), x(j+1, \ldots, x(j+i-1)$, respectively.
3. Divide each item of $x(n)$ by the square root of N to get the Hartley transform sequence. The resulting sequence $x(n)$ where $n = 0, 1, \ldots, N-1$ holds the transform.

A complete C++ source code corresponding to this algorithm is given in Fig. 9. The source code is general enough to handle any case for $N = 2^m$, where m is a positive integer.

Compared to the discrete Fourier transform, the discrete Hartley transform involves only real arithmetic and provides a real transform sequence. As a result, it requires less arithmetic and memory or storage space for computational pur-

```
// This C++ implementation of the fast Hartley
// transform is based on the algorithm
// proposed by H.S. Hou, IEEE Transactions on Computers,
// Vol. C-36, No. 2, pp. 147-156, February 1987.

// The input data file ''INPUT.DAT'' must contain the sample
// size, N, as the first data in the file. Then, the data
// sequence, with items separated by whitespaces, must
// follow in the file.
// The sample size, N, must be an integer power of 2.
// The transform sequence along with the sample size is
// saved in the output file ''OUTPUT.DAT''.

#include <fstream.h>
#include <math.h>
const double pi = 3.1415927;

void main()
{
  int N;
  double sqn;
  double *x, *y, *g;
  int n;

  ifstream infile;
  ofstream outfile;

  void FHT(double [], double [], double [], const int);

  // Read Data from file INPUT.DAT
  infile.open(''INPUT.DAT'');  // open file input file
  infile >> N;                 // read size, N
  x = new double[N];           // setup input data array
                               // with N items

  for (n = 0; n < N; ++ n)     // read data sequence
    infile >> x[n];
  infile.close();              // close input file

  // Setup auxiliary arrays
  y = new double[N];
  g = new double[N];

  // Perform fast Hartley transform
  FHT(y, g, x, N);

  // Release auxliary arrays
  delete [] y;
  delete [] g;

  // Scan transform sequence
  sgn = sqrt(N);
  for (n = 0; n < N; ++ n)
  x[n] = x[n] / sqn;

  // Save transform sequence
  outfile.open(''OUTPUT.DAT'');
  outfile << N << endl;                    // save size
  for (n = 0; n < N; ++ n)
     outfile << ''   '' << x[n] << endl;  // save transform
                                           // sequence
  outfile.close();                         // close output
                                           // file

  // Release the data array
  delete [] x;
}

void FHT (double y[], double g[], double x[], const int N)
{
 int i, j, m;

 void bit_reverse(double [], double [], const int);
 void RHT(double [], double [], const int);

 bit_reverse(y, x, N);

 for (i = 2; i <= N; i = i * 2)
 {
  for (j = 0; j < N; j = j + i)
  {
   for (m = j; m < j + i; ++m)  // copy i data items
    g[m−j] = x[m];              // of x in g

   RHT(y, g, i);                 // perform computation

   for (m = j; m < j + i; ++m)  // copy i calculated
                                // items
    x[m] = y[m−j];              // of y in x
  }
 }
}

// Recursive computation step
void RHT(double y[], double g[], const in M)
{
 int k;
 int L;
 double cfk, sfk;

 L = M >> 1;              // Divide M by 2 (L = M/2)

 y[0] = g[0] + g[L];
 y[L] = g[0] − g[L];

 for (k = 1; k < L; ++ k)
 {

  cfk = cos (2*pi*k/M);
  sfk = sin (2*pi*k/M);

  y[k] = g[k] + g[k + L] * cfk + g[M−k] * sfk;
  y[k + L] = g[k] − g[k + L] * cfk − g[M−k] * sfk;
 }
}

void bit_reverse(double y[], double x[], const int N)
{
  int i, incr, j;
  void arrange(double [], double [], int, int);

  // Bit reverse
  for (i = 1; i < N/2; i = i * 2)
  {
    incr = N/i;
    for (j = 0; j < N; j = j + incr)
      arrange(y, x, i, j + incr);
  }
}

void arrange(double y[], double x[], int first, int last)
{
 int mid, i, j;
 mid = (first + last)/2;
 for (i = first, j = first; i < mid; ++ i, j = j + 2)
   y[i] = x[j];

 for (i = mid, j = first + 1; i < last; ++ i, j = j + 2)
   y[i] = x[j];

 for (i = first; i < last; ++ i)
   x[i] = y[i];
}
```

Figure 9. A C++ program for fast Hartley transform.

poses. Also, for speed-critical, real-time applications, the hardware implementation of the discrete Hartley transform requires less hardware and is more efficient. These inherent advantages and the availability of the fast algorithms are the reasons why the Hartley transform is finding applications in many areas of science and engineering such as power engineering, data compression, speech coding, speech processing, image coding, image processing, optics, digital filtering, and biomedical engineering.

MULTIDIMENSIONAL HARTLEY TRANSFORM

The one-dimensional definition of the Hartley transform can easily be extended for multidimensional cases. Particularly, the two-dimensional Hartley transform for a function $h(x, y)$ can be described as

$$H(\alpha,\beta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\text{cas}[2\pi(\alpha x + \beta y)]\,dx\,dy \qquad (13)$$

and the corresponding two-dimensional inverse Hartley transform can be described as

$$h(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\alpha,\beta)\text{cas}[2\pi(\alpha x + \beta y)]\,d\alpha\,d\beta \qquad (14)$$

Properties of the two-dimensional Hartley transform can be obtained in the same way as for the one-dimensional case. Two-dimensional Hartley transform techniques are used in image processing as well as in analog and digital optical image processing. The two-dimensional discrete Hartley transforms and its inverse are given by

$$H(u,v) = \frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} h(m,n)\,\text{cas}\,2\pi\left(\frac{mu}{M} + \frac{nv}{N}\right) \qquad (15)$$

and

$$x(m,n) = \frac{1}{\sqrt{MN}}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} H(u,v)\,\text{cas}\,2\pi\left(\frac{mu}{M} + \frac{nv}{N}\right) \qquad (16)$$

The fast Hartley transform algorithm described in the earlier section for the one-dimensional case also can be used for this two-dimensional case if we manipulate Eq. (15). The kernel cas function in Eq. (15) is not separable. However, from Table 1, we can establish that

$$\begin{aligned}\text{cas}(\alpha+\beta) &= \cos(\alpha)\text{cas}(\beta) + \sin(\alpha)\text{cas}(-\beta)\\ &= \frac{1}{2}[\text{cas}(\alpha)\text{cas}(\beta) + \text{cas}(-\alpha)\text{cas}(\beta)\\ &\quad + \text{cas}(\alpha)\text{cas}(-\beta) - \text{cas}(-\alpha)\text{cas}(-\beta)]\end{aligned}$$

Accordingly, the kernel function $\text{cas}(2\pi(mu/M + nv/N))$ in Eq. (15) can be expanded, and hence $H(u, v)$ can be expressed as

$$H(u,v) = \frac{1}{2}[F(u,v) + F(-u,v) + F(u,-v) - F(-u,-v)]$$

where

$$F(u,v) = \frac{1}{\sqrt{MN}}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} x(m,n)\,\text{cas}\,2\pi\left(\frac{mu}{M}\right)\text{cas}\,2\pi\left(\frac{nv}{N}\right) \qquad (17)$$

Obviously, $F(u, v)$ is separable. Hence the one-dimensional fast transform algorithm can be used to compute $F(u, v)$, $F(-u, v)$, $F(u, v)$, and $F(-u, -v)$, and afterwards $H(u, v)$ can be obtained through simple addition and subtraction.

As stated earlier, the Hartley transform has many advantages over the Fourier transform, mainly because the Hartley transform is real for a real function or a real data sequence. It is computationally more efficient with respect to time and storage space. Additionally, for hardware implementation, the Hartley transform requires less hardware or VLSI area on the chip than the Fourier transform. An application that uses the Fourier transform can use the Hartley transform instead with some possible advantages.

Although the transform was introduced in 1942 by Hartley, it is R. N. Bracewell's 1983 work (2) and his other subsequent works that have brought attention to and popularized the Hartley transform. It has been found that the Hartley transform is very suitable for optical implementation because the transform representing the optical intensity is real for a real image (2). The Hartley transform has found many applications in science and engineering. The trend shows that the interest in the Hartley transform will continue in the future. The interest is evident from increasing number of publications on its theoretical development as well as on its applications every year.

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Dulal C. Kar
Virginia Polytechnic Institute and State University
V. V. Bapeswara Rao
North Dakota State University

HAZARDS, ELECTROLYTIC CELLS. See Electrolytic cell safety.

HDTV TRANSMITTERS. See Transmitters for digital television.

HEALTH CARE BY TELEVISION. See Telemedicine.

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