INTEGRAL EQUATIONS

Mathematics play a very important role in all the areas of electrical engineering. Whenever we are asked to develop a system or address a problem, the first thing we need to do is to develop a simple model. Many a times, this simple model turns out to be a mathematical model. The mathematical model lets us study many important aspects of the problem thoroughly and in an inexpensive manner.

In this article, we deal with an area of mathematics known as *integral equations*. We define an equation as an integral equation when the unknown quantity, i.e., the quantity to be determined, is under an integral sign. Integral equations are usually formulated when it is required to obtain the driving mechanism (input) of a physical system, given the description of the system along with the response function (output). For electrical engineers, the physical system may be an electrical circuit, an electrical machine, or, sometimes, a complex structure such as a fighter aircraft whose electromagnetic signature is the quantity of interest. Similarly, in many situations in electrical engineering, the response function may, simply, be the voltage at some given terminals or the current flowing in a wire.

There are several methods to solve integral equations (1) using complex mathematics. However, in many practical situations, these methods are inadequate and, quite often, we need to resort to numerical methods to solve these equations. In the following section, we formally introduce integral equations using simple mathematical language. We also introduce standard terminology to describe such equations and describe various types of integral equations. In the second section, we describe a general numerical method, known as method of moments, to solve these equations. In the third section, we present a new technique which makes the method of moments technique computationally more efficient along with a set of numerical results. Note that, although the topic of integral

J. Webster (ed.), *Wiley Encyclopedia of Electrical and Electronics Engineering*. Copyright © 1999 John Wiley & Sons, Inc.

equations is really a mathematical subject, we develop the subject by using examples from electrical engineering and in fact, from electromagnetic theory. It must be clearly understood that this way of treatment of the subject does not necessarily preclude the application of the techniques discussed in this article into other areas of engineering.

INTEGRAL EQUATIONS

Mathematically speaking, an equation involving the integral of an unknown function of one or more variables is known as integral equation. One of the most common integral equations encountered in electrical engineering is the *convolution integral* given by

$$\int X(\tau)H(t,\tau)d\tau = Y(t) \tag{1}$$

In Eq. (1), we note that the response function $Y(t)$ and the system function $H(t, \tau)$ is known and we need to determine the input $X(\tau)$. Of course, if $X(\tau)$ and $H(t, \tau)$ are known and we need to determine $Y(t)$, then Eq. (1) simply represents an integral relationship which can be performed in a straightforward manner. We further note that $H(t, \tau)$ is also commonly known as *impulse response* if Eq. (1) represents the system response of a linear system. In general, in mathematics and in engineering literature, $H(t, \tau)$ is known as *Green's function* or *kernel function*. We also acknowledge that, for some other physical systems, $Y(t)$ and $X(t)$ may represent the driving force and response functions, respectively.

Next, we note that Eq. (1) is known as integral equation of first kind. We also have another type of integral equation given by

$$C_1X(t) + C_2\int X(\tau)H(t,\tau)d\tau = Y(t) \tag{2}$$

where C_1 and C_2 are constants.

In Eq. (2), we note that the unknown function $X(t)$ appears both inside and outside the integral sign. Such equation is known as the integral equation of second kind. Further, we also see in electrical engineering yet another type of integral equation given by

$$C_1\int X(\tau)H(t,\tau)d\tau + C_2X(t) + C_3\frac{dX(t)}{dt} = Y(t) \tag{3}$$

which is known as *integro-differential equation.*

It may be noted that for a limited number of kernel and response functions, in Eqs. (1–3), it is possible to obtain the solution using analytical methods. Several textbooks have been written to discuss the mathematical aspects of the integral equations from an analytical point of view (2–4). However, for a majority of practical problems, these equations can be solved using numerical methods only. Fortunately, in this day and age, we can obtain very accurate numerical solutions owing to the availability of fast digital computers. In the following section, we discuss a general numerical technique, popularly known as *method of moments,* to solve the integral Eqs. (1–3).

METHOD OF MOMENTS SOLUTION

The method of moments (MoM) solution procedure was first applied to electromagnetic scattering problems by Harrington (5). Consider a linear operator equation given by

$$AX = Y \tag{4}$$

where A represents the integral operator, Y is the known excitation function, and X is the unknown response function to be determined. Now, let X be represented by a set of known functions, termed as basis functions or expansion functions, (p_1, p_2, p_3, . . .,) in the domain of A as a linear combination:

$$X = \sum_{i=1}^{N} \alpha_i p_i \tag{5}$$

where α_i values are scalars to be determined. Substituting Eq. (5) into Eq. (4), and using the linearity of A, we have

$$\sum_{i=1}^{N} \alpha_i A p_i = Y \tag{6}$$

where the equality is usually approximate. Let (q_1, q_2, q_3, . . .) define a set of testing functions in the range of A. Now, multiplying Eq. (6) with each q_j and using the linearity property of the inner product, we obtain

$$\sum_{i=1}^{N} \alpha_i \langle q_j, Ap_i\rangle = \langle q_j, Y\rangle \tag{7}$$

for $j = 1, 2, . . ., N$. The set of linear equations represented by Eq. (7) may be solved using simple matrix methods to obtain the unknown coefficients α_i.

The simplicity of the method lies in choosing the proper set of expansion and testing functions to solve the problem at hand. Further, the method provides a most accurate result if properly applied. However, for the integral equation operators, the method generates a dense matrix which may be expensive in terms of computer storage requirements when complex systems are involved. In the following subsections, we discuss the application of the method of moments to some commonly used integral equations in engineering and science.

Integral Equations without Derivatives

In this section, we develop simple numerical methods to solve integral equations (both first and second kind) applying the method of moments. Further, we restrict our treatment to integral equations with single independent variable (one-dimension) only. The extension to multiple variables is straightforward and hence is not considered here. The numerical methods are general methods, and thus applicable to a variety of practical problems.

Consider an integral equation given by

$$\int_{x'=-w}^{w} u(x')g(x,x')dx' = f(x) \qquad x \in (-w, w) \tag{8}$$

in which $u(x)$ is the unknown function to be determined. For the method of moments analysis of such problems, we develop a numerical scheme known as collocation method, subdomain

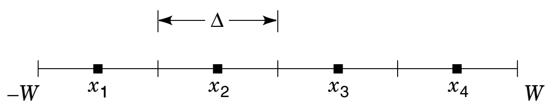

Figure 1. Match points for the integral equation.

method, or point matching method. For this procedure, we first divide the interval $-w$ to w into N equal segments of width Δ as shown in Fig. 1.

The segment center points are given by

$$x_i = -w + 0.5(i-1)\Delta \qquad i = 1, 2, \ldots, N \tag{9}$$

Note that while defining Eq. (9), we have divided the interval $-w$ to w into equal segments, although this need not be the case in general.

The next step in the method of moments solution procedure is to define a suitable set of basis and testing functions. Our research shows that, for this type of problem, i.e., the integral equations with no derivatives, the most convenient and simple set of functions are pulse functions with unit amplitude as basis functions and Dirac delta distributions (functions) as testing functions. In the following, we formally define these functions, as shown in Fig. 2, given by

$$p_i(x) = \begin{cases} 1 & x_i - \dfrac{\Delta}{2} \le x \le x_i + \dfrac{\Delta}{2} \\ 0 & \text{Otherwise} \end{cases} \tag{10}$$

and

$$q_j(x) = \delta(x - x_j) \tag{11}$$

Here, we emphasize that Eqs. (10) and (11) are by no means the only set of functions used in practice. It is quite possible to define a completely different set of functions as long as these functions satisfy a certain set of conditions (6–8). Further, it is also possible to carry-out an entirely different scheme in which the expansion and testing functions are defined over the whole interval without ever dividing the solution region into subsections. Such numerical schemes are known as *entire domain methods.* Entire domain methods are known to be mathematically unstable (5), which may be overcome by a suitable choice of testing and basis functions or a combination of subdomain/entire domain functions (9). However, we will not present the numerical treatment with entire domain functions in this work since the subject is still in research stage.

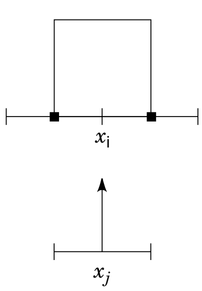

Figure 2. Pulse function and delta function.

First, we shall consider the testing procedure. Here, we multiply the Eq. (8) by the testing function q_j and integrate over the whole interval to obtain a set of equations given by

$$\int_{x'=-w}^{w} u(x')g(x_j, x')dx' = f(x_j) \qquad j = 1, 2, \ldots N \tag{12}$$

Observe that, while evaluating Eq. (12), we made use of the well-known properties of the delta distribution (function). Also note that Eq. (12) is actually a set of N equations for each j, and x_j represents the value of the independent variable at the center of the jth subdomain. Further, observe that we are matching the left and right hand sides of Eq. (12) at points x_j for $j = 1, 2, \ldots, N$. Thus, these points are known also as *match points.*

Next we consider the expansion procedure. By using the basis functions defined in Eq. (10), the unknown quantity $u(x)$ may be written as

$$u(x) = \sum_{i=1}^{N} \alpha_i p_i \tag{13}$$

where α's represent the unknown scalar coefficients. Substituting Eq. (13) into Eq. (12), we have

$$\sum_{i=1}^{N} \alpha_i \int_{x'=x_i-\Delta/2}^{x_i+\Delta/2} g(x_j, x')dx' = f(x_j) \qquad j = 1, 2, \ldots N \tag{14}$$

Note that, Eq. (14) may be written as a matrix equation, given by

$$[Z][I] = [V] \tag{15}$$

where

$$Z_{ji} = \int_{x'=x_i-\Delta/2}^{x_i+\Delta/2} g(x_j, x')dx' \tag{16}$$

$$V_j = f(x_j) \tag{17}$$

and the column vector $[I]$ contains unknown coefficients α's. Except for certain special cases, the matrix $[Z]$ is a well-conditioned matrix and hence the solution of Eq. (15) is straightforward. Also, the integrations involved in Eq. (16) may be either performed analytically or numerically depending on the exact nature of the kernel function.

Lastly, the numerical method described so far is also known as pulse expansion and point matching method. In the following, we present an example problem based on the procedure described so far.

Example. Consider an infinitely long conducting strip of width of 0.1 m located symmetrically at the origin as shown

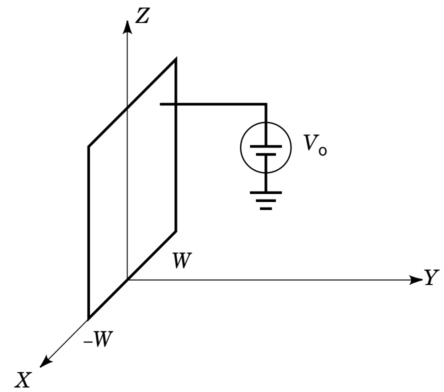

Figure 3. Infinite strip raised to 1 V potential.

in Fig. 3. The strip is raised to a potential of 1 V. Note that the reference point (i.e., $V = 0$) is at $x = 1$ m. Calculate the charge distribution on the strip.

SOLUTION. Following the basic principles of electrostatics, an integral equation may be developed, given by

$$\int_{x'=-0.05}^{0.05} q_s(x') \ln|x - x'| dx' = 2\pi\epsilon_0 \qquad x \in (-0.05, 0.05) \quad (18)$$

where $\epsilon_0 = 8.854e - 12$ is the permittivity of the surrounding medium. Following the numerical procedures described so far, we obtain the elements of the $[Z]$-matrix given by

$$\begin{aligned} Z_{ji} &= \int_{x'=x_i-\Delta/2}^{x_i+\Delta/2} \ln|x_j - x'| dx' \\ &= \Delta - \frac{\Delta}{2} \ln|(x_j - x_i)^2 - (\Delta/2)^2| \\ &\quad - (x_j - x_i) \ln \frac{|x_j - x_i + \Delta/2|}{|x_j - x_i - \Delta/2|} \end{aligned} \quad (19)$$

and the elements of $[V]$-matrix are

$$V_j = 2\pi\epsilon_0 \quad (20)$$

In Fig. 4, we present the charge distribution for N equal to 10, 50, and 100 obtained by solving the integral Eq. (18).

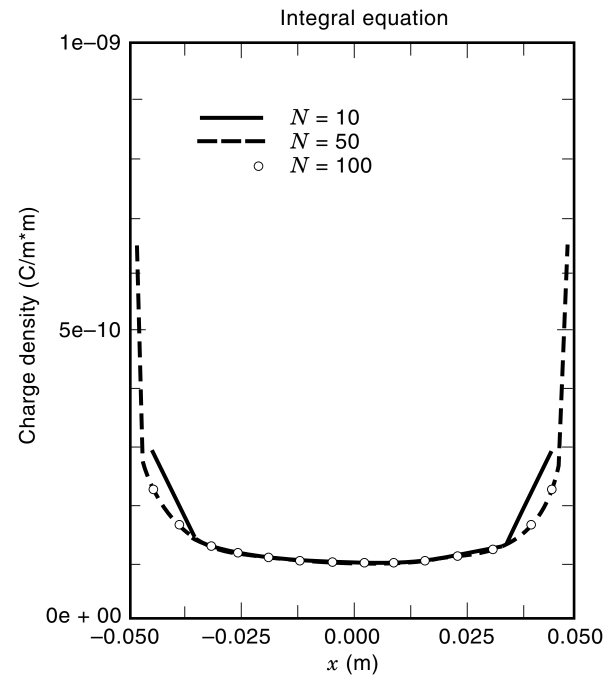

Figure 4. Charge density distribution on the infinite strip.

Integral Equations with Derivatives

In this section, we develop simple numerical methods to solve integrodifferential equations, i.e., integral equations with derivative operators, applying the method of moments. As before, we restrict our treatment to integral equations with a single independent varaible (one-dimension) only. The extension to multiple variables is straightforward and hence is not considered here. The numerical methods are general methods, and thus applicable to a variety of practical problems.

We consider two cases in this section: the first-order integrodifferential equation, and the second-order integrodifferential equation. Obviously, higher order derivatives may be handled in a similar manner.

First Order Integrodifferential Equation. Consider a first-order integrodifferential equation given by

$$\frac{\partial}{\partial x} \int_{x'=-w}^{w} u(x') g(x, x') dx' = f(x) \qquad x \in (-w, w) \quad (21)$$

subject to

$$\int_{x=-w}^{w} u(x) dx = 0 \quad (22)$$

The Eq. (22) is also known as a *constraining equation*. In a variety of situations, constraining equations can be implicitly enforced by a proper choice of basis or testing functions. This necessitates a more elaborate construction of basis/testing functions which, although it seems to be complicated, results in an efficient numerical solution. It is quite easy to see that a straightforward application of the method discussed in the previous section, i.e., pulse-expansion and point matching method, results in $N \times N$ matrix. However, the application of the constraint equation adds one more column to the $[Z]$-matrix, thus making the problem over-determined system. Further, other numerical problems, such as stability and nonuniqueness, set in when MoM is applied blindly. Thus, we develop the following numerical procedure for this case.

As before, the interval $(-w, w)$ is divided into N equal segments. But for this case, the match points are labled in the following way for mathematical convenience as shown in Fig. 5.

$$x_i = -w + i \times \Delta \qquad i = 1, 2, \ldots, N - 1 \quad (23)$$

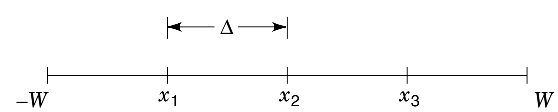

Figure 5. Match points for integrodifferential equation.

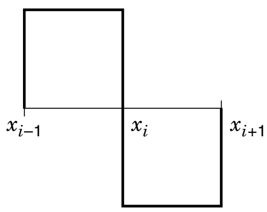

Figure 6. Pulse-doublet function.

In order to enforce the constraining Eq. (22), we let the basis function to overlap over two subdomains with positive unit height in the first subdomain and negative unit pulse in the second subdomain, as shown in Fig. 6.

Thus, mathematically, we define the basis function as

$$p_i(x) = \begin{cases} 1 & x_{i-1} \le x \le x_i \\ -1 & x_i \le x \le x_{i+1} \\ 0 & \text{Otherwise} \end{cases} \tag{24}$$

and express the unknown quantity $u(x)$ as

$$u(x) = \sum_{i=1}^{N-1} \alpha_i p_i \tag{25}$$

Notice that, by defining basis functions as in Eq. (24), Eq. (22) is automatically satisfied, which can be proved as

$$\begin{aligned} \int_{x=-w}^{w} u(x)dx &= \sum_{i=1}^{N} \alpha_i \int p_i dx \\ &= \sum_{i=1}^{N} \alpha_i \left[\int_{x_{i-1}}^{x_i} dx - \int_{x_i}^{x_{i+1}} dx \right] \\ &= 0 \end{aligned} \tag{26}$$

The functions defined by Eq. (24) are known as *pulse doublet functions*.

Next, we define the testing procedure for this case. Notice that we have one derivative on the integral sign. By simple mathematical manipulation, we transform the derivative operator onto the testing function q_j. By using a compact notation

$$\langle f, g \rangle = \int fg dx \tag{27}$$

we can write the integrodifferential Eq. (21) as

$$\left\langle \frac{\partial v}{\partial x}, q_j \right\rangle = \langle f(x), q_j \rangle \tag{28}$$

where

$$v(x) = \int_{x'=-w}^{w} u(x')g(x, x')dx' \tag{29}$$

Then, we have

$$\begin{aligned} \left\langle \frac{\partial v}{\partial x}, q_j \right\rangle &= \int \frac{\partial v}{\partial x} q_j dx \\ &= [q_j v] - \int \frac{\partial q_j}{\partial x} v dx \end{aligned} \tag{30}$$

The first term in Eq. (30) can be set to zero if $q_j = 0$ at the ends of the subdomain.

Keeping this procedure in mind, we select the testing functions in such a way that when the derivative is transformed onto the testing function the result must be a delta distribution (function). A unit pulse function, as shown in Fig. 7, has this property whose derivative happens to be two delta distributions on either end of the pulse.

Thus, for first-order integrodifferential equations, we choose the testing function q_j as

$$q_j(x) = \begin{cases} 1 & x_j - \dfrac{\Delta}{2} \le x \le x_j + \dfrac{\Delta}{2} \\ 0 & \text{Otherwise} \end{cases} \tag{31}$$

The numerical procedure may be best illustrated by the following example.

Example. Consider that an infinitely long conducting strip of width 1 m, as shown in Fig. 3, is immersed in an electrostatic field. Calculate the charge distribution on the strip.

SOLUTION. Following the basic principles of electrostatics, and applying the electric field boundary condition on perfect conducting bodies, an integral equation may be developed, given by

$$\frac{\partial}{\partial x} \int_{x'=-w}^{w} q_s(x') \ln|x - x'| dx' = 2\pi\epsilon \boldsymbol{a}_x \cdot \boldsymbol{E}^{\mathrm{i}} \qquad x \in (-w, w) \tag{32}$$

subject to

$$\int_{x=-w}^{w} q_s(x)\, dx = 0 \tag{33}$$

where $\boldsymbol{E}^i$, q_s, and $\boldsymbol{a}_x$ are the impressed electric field, charge density, and the x-directed unit vector, respectively. For the numerical solution, we divide the interval $(-w, w)$ into N subdomains of width Δ and label the match points as shown in Fig. 5. Notice that when the interval is divided into N divisions, we actually have $N - 1$ match points.

Defining the testing functions by Eq. (31), and carrying out the mathematical steps outlined in Eq. (30), we get

$$\begin{aligned} &\int_{x'=-w}^{w} q_s(x') \ln\left| x_j + \frac{\Delta}{2} - x' \right| dx' \\ &\quad - \int_{x'=-w}^{w} q_s(x') \ln\left| x_j - \frac{\Delta}{2} - x' \right| dx' \\ &\quad = 2\pi\epsilon\Delta \boldsymbol{a}_x \cdot \boldsymbol{E}^{\mathrm{i}}(x_j) \end{aligned} \tag{34}$$

for $j = 1, 2, \ldots, N - 1$.

Next, we apply the expansion procedure. By selecting the basis functions as described in Eq. (25), the constraining Eq.

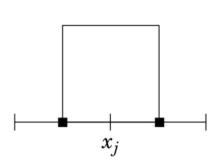

Figure 7. Pulse testing function.

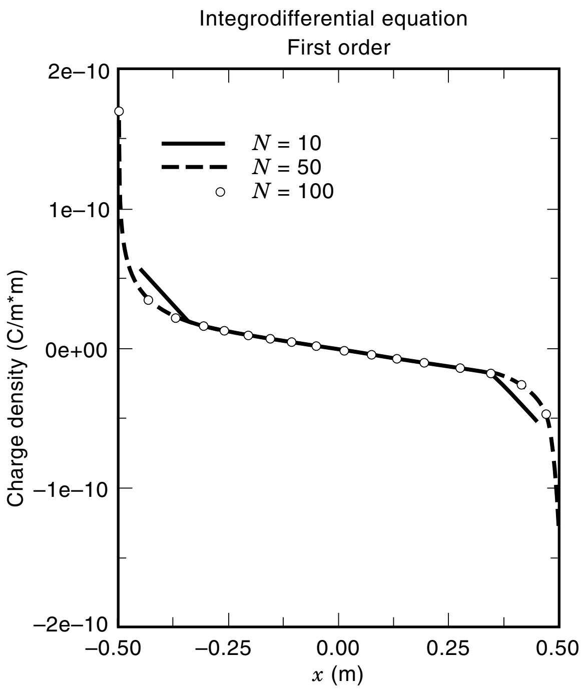

Figure 8. Charge density distribution on the infinite strip immersed in electric field $\boldsymbol{E}^i = \boldsymbol{a}_x$.

(33) is automatically enforced. Thus, applying the method of moments procedure, we obtain $[Z][I] = [V]$, where

$$\begin{aligned} Z_{ji} = & \int_{x_{i-1}}^{x_i} \ln\left|x_j + \frac{\Delta}{2} - x'\right| dx' \\ & - \int_{x_i}^{x_{i+1}} \ln\left|x_j + \frac{\Delta}{2} - x'\right| dx' \\ & - \int_{x_{i-1}}^{x_i} \ln\left|x_j - \frac{\Delta}{2} - x'\right| dx' \\ & + \int_{x_i}^{x_{i+1}} \ln\left|x_j - \frac{\Delta}{2} - x'\right| dx' \end{aligned} \tag{35}$$

and

$$V_j = 2\pi\epsilon\Delta \boldsymbol{a}_x \cdot \boldsymbol{E}^{\mathrm{i}}(x_j) \tag{36}$$

In Fig. 8, we present the charge distribution for N equal to 10, 50, and 100 obtained by solving the integrodifferential Eq. (32). Notice that, in this procedure, the dimension of the system matrix is $N - 1$.

Second-Order Integrodifferential Equation. In this section, we consider techniques for solving the integrodifferential equation

$$\frac{\partial^2}{\partial x^2}\int_{x'=-w}^{w} u(x')g(x,x')\,dx' = f(x) \qquad x \in (-w, w) \tag{37}$$

where the unknown function $u(x)$ must satisfy the boundary conditions

$$u(w) = u(-w) = 0$$

These types of integral equations usually appear in electromagnetic and acoustic scattering problems, the most common being the dipole antenna problem in antenna engineering. Further, the treatment of second-order integrodifferential equation, coupled with the treatment of first-order derivatives, provides a solution procedure for handling higher order derivatives.

We begin our analysis by rewriting the integrodifferential Eq. (37) in the following form:

$$\frac{\partial}{\partial x}\int_{x'=-w}^{w} u(x')\frac{\partial g(x,x')}{\partial x}\,dx' = f(x) \qquad x \in (-w, w) \tag{38}$$

For almost all mathematical problems in engineering, there exists a definite relationship between $\partial g/\partial x$ and $\partial g/\partial x'$. In fact, for electromagnetic (EM) and acoustic scattering problems, we have $\partial g/\partial x = -\partial g/\partial x'$. Using this relationship, we can write Eq. (38), at least for EM and acoustic problems, as

$$\frac{\partial}{\partial x}\int_{x'=-w}^{w} \frac{\partial u(x')}{\partial x'}g(x,x')\,dx' = f(x) \qquad x \in (-w, w) \tag{39}$$

Now, we have an integrodifferential equation of first order which we already know how to handle. At first, we divide the interval $(-w, w)$ into N segments and label $N - 1$ match points as shown in Fig. 5. The definition of testing functions and the testing procedure is identical to the case of first-order integrodifferential equation and hence need not be repeated again. However, we need to look more closely at the basis functions.

Note that, for the case of first-order integrodifferential equations, we defined the pulse doublet as the expansion function and obtained the solution for the unknown function. In the present case we can do the same thing, if we define the antiderivative of pulse doublet as the expansion function. Following this logic, we define the basis functions for the solutions of second-order integrodifferential equation as

$$p_i(x) = \begin{cases} 1 - \dfrac{x_i - x}{\Delta} & x_{i-1} \le x \le x_i \\ 1 + \dfrac{x_i - x}{\Delta} & x_i \le x \le x_{i+1} \\ 0 & \text{Otherwise} \end{cases} \tag{40}$$

The functions described in Eq. (40), and shown in Fig. 9, are popularly known as *Triangle functions,* which are linear piece-wise.

Thus, for the solution of second-order integrodifferential equations, we employ triangle function expansion and pulse function testing. We describe the numerical procedure using the following example.

Example. Consider a finite-length straight wire, radius $a = 0.001\lambda$, and length $2h = 0.5\lambda$ illuminated by an electromag-

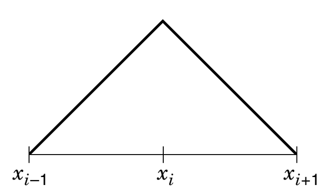

Figure 9. Triangle basis function.

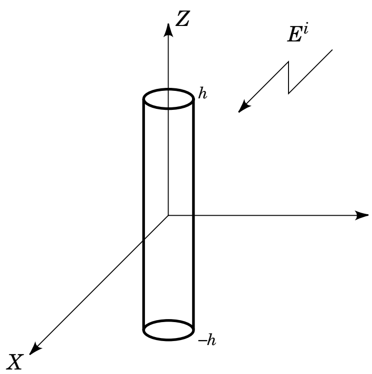

Figure 10. Straight wire illuminated by a plane wave.

netic plane wave (wave length λ) as shown in Fig. 10. Calculate the current induced on the wire.

SOLUTION. Since the radius a is very small compared to λ and h we can use the thin-wire theory (10) to formulate the integrodifferential equation. Following the mathematical procedures described in (11), we derive the following integral equation, given by

$$\frac{\partial}{\partial z}\int_{z'=-h}^{h}\frac{\partial I(z')}{\partial z'}G(z-z')\,dz' + k^2\int_{z'=-h}^{h} I(z')G(z-z')dz' = -j\frac{4\pi k}{\eta}E_z^i(z) \qquad z\in(-h,h) \quad (41)$$

where

$$G(z-z') = \frac{e^{-jkR}}{R} \quad (42)$$

and

$$R = \sqrt{(z-z')^2 + a^2} \quad (43)$$

In Eqs. (41–43), I is the unknown current induced on the wire, $E_z^i(z)$ is the z-component of the incident plane wave, $k = 2\pi/\lambda$ is the wave number, and η is the wave impedance of the surrounding medium.

First of all, divide the wire region $(-h, h)$ into N equal segments labeling $N - 1$ match points as shown in Fig. 5. Next, for this problem, we choose the expansion functions p_i defined in Eq. (40) to express the unknown current I and the testing functions q_j defined in Eq. (31).

Thus, we have

$$I = \sum_{i=1}^{N-1}\alpha_i p_i \quad (44)$$

Next, we consider the testing procedure. By following the same procedures of the previous section on first-order integrodifferential equations, the testing procedure yields,

$$\int_{z'=-h}^{h}\frac{\partial I(z')}{\partial z'}G\left(z_j+\frac{\Delta}{2}-z'\right)dz' - \int_{z'=-h}^{h}\frac{\partial I(z')}{\partial z'}G\left(z_j-\frac{\Delta}{2}-z'\right)dz' + \Delta k^2\int_{z'=-h}^{h} I(z')G(z_j-z')dz' = -j\frac{4\pi k\Delta}{\eta}E_z^i(z_j) \quad (45)$$

for $j = 1, 2, \ldots, N - 1$. Notice that, in Eq. (45), the integrations on the second term and the right hand side of the Eq. (41) are approximated by a simple one-point rule.

Substituting the expansion Eq. (44) into Eq. (45), we obtain the matrix equation $[(1/\Delta)[Z^a] + (k^2\Delta)[Z^b]]\,[I] = [V]$ where the matrix elements are:

$$\begin{aligned} Z_{ji}^a = &\int_{z_{i-1}}^{z_i} G\left(z_j+\frac{\Delta}{2}-z'\right)dz' \\ &- \int_{z_i}^{z_{i+1}} G\left(z_j+\frac{\Delta}{2}-z'\right)dz' \\ &- \int_{z_{i-1}}^{z_i} G\left(z_j-\frac{\Delta}{2}-z'\right)dz' \\ &+ \int_{z_i}^{z_{i+1}} G\left(z_j-\frac{\Delta}{2}-z'\right)dz' \end{aligned} \quad (46)$$

$$\begin{aligned} Z_{ji}^b = &\int_{z_{i-1}}^{z_i}\left\{1-\frac{z_i-z}{\Delta}\right\}G(z_j-z')dz' \\ &+ \int_{z_i}^{z_{i+1}}\left\{1+\frac{z_i-z}{\Delta}\right\}G(z_j-z')dz' \end{aligned} \quad (47)$$

and

$$v_j = -j\frac{4\pi k\Delta}{\eta}E_z^i(z_j) \quad (48)$$

The integrations involved in Eqs. (46) and (47) may be carried out using the methods discussed in (12).

In Fig. 11, we present the current induced on a half-wave dipole wire scatterer due to a unit-amplitude, normally incident plane wave for N equal to 20 and 50 divisions obtained by using Eqs. (46–48).

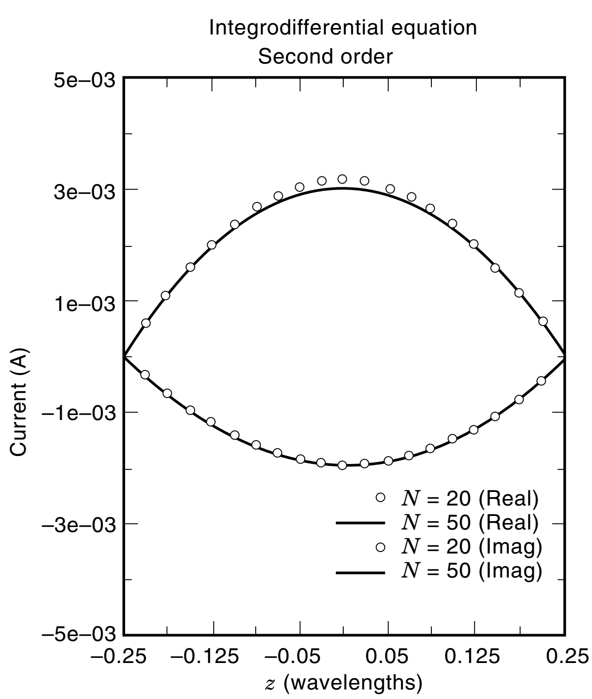

Figure 11. Current induced on the wire scatterer.

Integral Equations with More Variables

In the previous subsection, we discussed numerical methods applying the method of moments to handle integral and integrodifferential equations with one independent variable. Extension to multivariable case is straightforward and follows the same numerical procedures discussed so far. For example, for two-variable case, i.e. $x - y$ plane, the solution region may be divided into square or rectangular cells and one can construct the basis and testing functions using the methods discussed in the previous section (13). For a more general situation, the solution region can be divided into triangular subdomains along with suitable basis and testing functions (14,15). Efficient solutions have been obtained for very complex problems using these methods in electromagnetics and acoustics (16–19) and it is quite possible that these methods found applications in other areas of engineering. Lastly, we mostly discussed only boundary-value problems in this work but solutions have also been obtained for initial value problems (20–28) using the same methods. An extensive application of method of moments to electromagnetic scattering problems may be obtained in (29).

SPARSE MATRIX METHODS

One major problem with MoM is the generation of a dense matrix and for complex problems, the dimension of this matrix can be prohibitively large. Usually, for electromagnetic and acoustic scattering problems, it is necessary to divide the solution region into small enough subdomains in order to obtain accurate results. By "small enough," we mean about 200 to 300 subdomains per square wavelength. In usual practice, we may typically solve for several thousand unknowns for large, complex problems. Quickly, this requirement becomes expensive in terms of computational resources and may even become impossible to handle. Hence, we look for alternate schemes to reduce the computational resources by generating a sparse matrix instead of a full matrix.

The generation of a sparse matrix in the method of moment solution procedure may be achieved in two ways: (a) by defining a special set of basis functions to represent the unknown quantity or (b) by handling the influence of the kernel function in a novel way. The usage of well-known, wavelet-type basis functions to provide the required sparsity belongs to the former category (30) and the application of fast multipole method (FMM) belongs to the latter category (31). So far, the wavelet-type basis functions have been applied to integral equations with one variable only, and it remains to be seen how these functions can be utilized for two or more variable cases. In contrast, in the FMM scheme, the matrix-vector product is carried out in a novel way and seems to work well for more complex problems. Unfortunately, the FMM is a complicated scheme and any reasonable summary of the method is beyond the scope of the present article.

There is yet another scheme, known as impedance matrix localization (IML), which achieves modest sparsity for simple problems (32). Notice that the kernel function is, in general, a decaying function with respect to the distance between the source and observation points. Thus, with increasing distances, the influence of a given source becomes negligible at a sufficiently distant observation point and may be actually set to zero. The IML scheme cleverly exploits this fact. However, there is a certain degree of arbitrariness in this scheme and it seems to work for simple problems only.

Recently a new method, known as generalized sparse matrix reduction scheme (GSMR), is proposed, which seems to improve on the IML method.

The basic concept utilized in the GSMR technique may be qualitatively illustrated as follows. Following similar procedures of the MoM, a moment matrix is also generated in the GSMR method. However, in contrast to the conventional moment method where interaction is computed from each and every cell on other cells, only the interaction from the self-cell and few neighboring cells is computed in the GSMR technique. In fact, for single variable problems (wire scatterer and two-dimensional, infinite cylinders) only the self-term and two neighboring terms on either side of the self-cell are generated in this technique. This implies that the moment matrix for the GSMR technique is essentially sparse. Further, the effect of nonself terms is taken into account by defining a set of linearly independent functions over the entire structure. In the mathematical terms the procedure, for single variable problems, may be described as follows:

Let $[Z]$ represent the moment matrix for a given problem generated by using appropriate basis and weighting functions. Note that, for well-defined problems with proper choice of basis and testing functions, the moment matrix is well-conditioned and diagonally strong. The jth row of the moment matrix may be written as

$$\sum_{i=1}^{N} Z_{j,i} I_i = V_j \qquad (49)$$

where all the matrix elements $Z_{j,i}$ are nonzero. In the new GSMR technique, the jth row is modified as

$$\sum_{i=j-1}^{j+1} \alpha_{j,i} Z_{j,i} I_i = \Gamma_j V_j \qquad (50)$$

where $\alpha_{j,j-1}$, $\alpha_{j,j}$, $\alpha_{j,j+1}$, and Γ_j are the unknown coefficients and the rest of terms in the row are set to zero. Further, dividing by $Z_{j,j}$, Eq. (50) may be rewritten as

$$\sum_{i=j-1}^{j+1} \beta_{j,i} I_i = \gamma_j V_j \qquad (51)$$

which may be written, using the matrix notation, as

$$[\boldsymbol{\beta}][I] = [V] \qquad (52)$$

where $[\boldsymbol{\beta}]$ is a sparse matrix with, at most, three nonzero elements per row.

Upon a close examination of Eq. (52), it is obvious that one needs to reconstruct the $[\boldsymbol{\beta}]$-matrix. This task may be accomplished by first setting $\gamma_j = 1$ for $j = 1, \ldots, N$ in Eq. (51).

Next, define three linearly independent functions, $I^{(1)}$, $I^{(2)}$, and $I^{(3)}$, over the entire domain of the problem. These functions may be thought of as source distributions. For the examples we discuss below, these functions are assumed to be a constant, $\cos(kl)$ and $\sin(kl)$ where $k = 2\pi/\lambda$ is the wave number and l is the parameter measured along the length of the independent variable in the integral equation.

The next step in the GSMR technique is to compute the corresponding response functions, $V^{(1)}$, $V^{(2)}$, and $V^{(3)}$. This task may be easily accomplished by using the assumed source distributions $I^{(1)}$, $I^{(2)}$, and $I^{(3)}$, and utilizing the Green's function for the problem.

Once we have $I^{(1)}$, $I^{(2)}$, $I^{(3)}$, $V^{(1)}$, $V^{(2)}$, and $V^{(3)}$, the $[\boldsymbol{\beta}]$-matrix may be constructed as follows:

- For any j, sample $I^{(1)}$, $I^{(2)}$, and $I^{(3)}$ at locations $j-1$, j, and $j+1$, and sample $V^{(1)}$, $V^{(2)}$, and $V^{(3)}$ at location j, and write the following system of equations:

$$\begin{aligned}\beta_{j,j-1}I_{j-1}^{(1)} + \beta_{j,j}I_{j}^{(1)} + \beta_{j,j+1}I_{j+1}^{(1)} &= V_{j}^{(1)}\\ \beta_{j,j-1}I_{j-1}^{(2)} + \beta_{j,j}I_{j}^{(2)} + \beta_{j,j+1}I_{j+1}^{(2)} &= V_{j}^{(2)}\\ \beta_{j,j-1}I_{j-1}^{(3)} + \beta_{j,j}I_{j}^{(3)} + \beta_{j,j+1}I_{j+1}^{(3)} &= V_{j}^{(3)}\end{aligned} \quad (53)$$

- Solve Eq. (53) to obtain $\beta_{j,j-1}$, $\beta_{j,j}$, and $\beta_{j,j+1}$ and store in the jth row of the $[\boldsymbol{\beta}]$-matrix.
- Repeat the previous two steps for all values of j.

Further, note that for $j = 1$ and $j = N$, we select $\beta_{1,N}$, $\beta_{1,1}$, and $\beta_{1,2}$, and, $\beta_{N,N-1}$, $\beta_{N,N}$, and $\beta_{N,1}$, respectively.

Once all the coefficients for each row are computed, we have successfully generated the new matrix representation for the integral equation. Finally, Eq. (52) may be solved efficiently using iterative methods such as the conjugate gradient method (32) or the GMRES method (33) since we are dealing with sparse matrices.

Example. Consider a 10λ straight wire, 0.001λ radius, illuminated by a normally incident plane wave. The matrix size for the MoM and GSMR method is 149×149 and 149×3, respectively. The results are shown in Fig. 12 and the comparison is excellent.

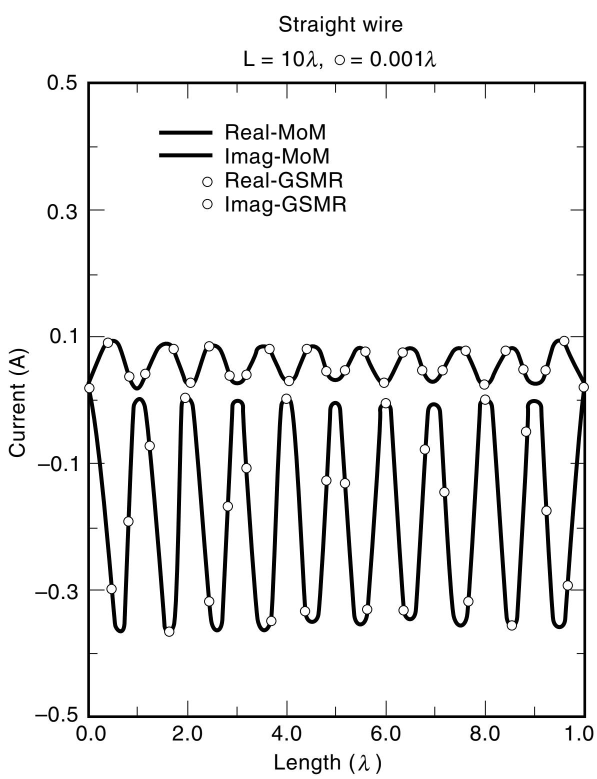

Figure 12. Current induced on the 10λ wire scatterer.

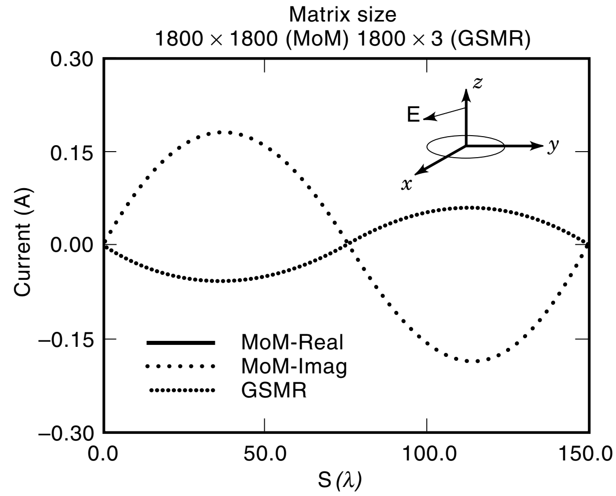

Figure 13. Current induced on the circular loop.

Example. Consider the case of a circular loop located in the $z = 0$ plane with center at the origin. The loop is illuminated by an x-polarized plane wave traveling along the z-axis. Figure 13 shows the results for $ka = 150$ where k and a are the wave number and the radius of the loop, respectively. The matrix size for the MoM and the GSMR technique 1800 is $\times$ 1800 and 1800×3, respectively. It is evident from the figure that the results compare very well with each other. This example clearly illustrates the applicability of the GSMR method for truly large bodies.

Example. Lastly, we present the case of an infinitely long, conducting strip illuminated by a transverse magnetic (TM) incident electromagnetic plane wave. The derivation of the governing integral equation for this problem may be found in (5). Figure 14 shows the current density induced on a 150λ bent strip obtained by applying MoM and GSMR techniques.

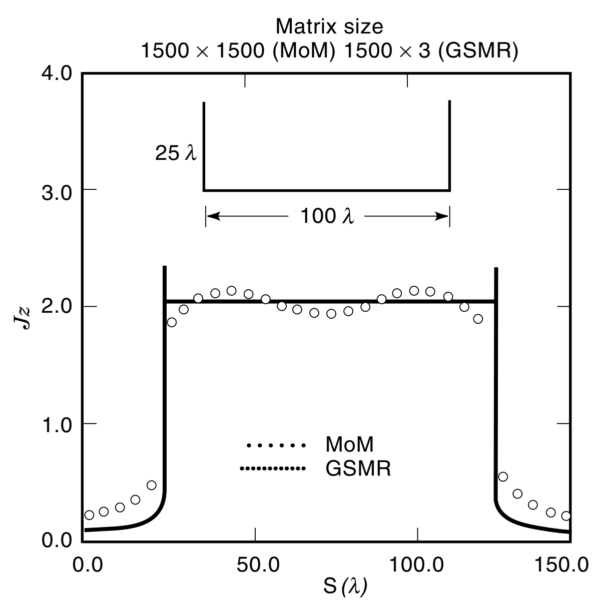

Figure 14. Current induced on the conducting bent strip by a TM incident plane wave. The cross-section of the strip is shown in the inset.

The comparison between both methods is reasonably accurate for both cases as evident from the figure.

Lastly, before closing the discussion on GSMR technique, the existence of $\boldsymbol{\beta}$ matrix may be explained in the following way. It may be noted that the moment matrix generated in the conventional MoM solution procedure is a representation of the unique relationship that exists between the source and the response. This relationship is specified in mathematical terms via the Green's function along with the boundary conditions. Further, this relationship holds for any source distribution and response function as long as the response function is derived utilizing the Green's function satisfying the appropriate boundary conditions. Since $\boldsymbol{\beta}$ matrix is developed using this unique relationship, Eq. (52) must represent the discretized form of the operator equation. Further, it should be noted that, although the operator equation is unique, the matrix representation is not necessarily unique. This is quite obvious since different basis and testing functions result in a different matrix representation. Also, one can perform elementary row and column operations on the given system of equations and arrive at another representation of the same operator equation.

However, as a word of caution, it may be noted that the GSMR technique is a recent concept and tested only on some simple problems. It is obvious that the procedure needs to be validated for more complex geometries. Presently, work is in progress to apply the GSMR technique to some of these cases.

BIBLIOGRAPHY

1. W. Pogorzelski, *Integral Equations and Their Applications,* Vol. I-III, New York: Pergamon, 1966.
2. B. L. Moiseiwitsch, *Integral Equations,* New York: Longman Inc., 1977.
3. A. C. Pipkin, *A Course on Integral Equations,* New York: Springer-Verlag, 1991.
4. I. Stakgold, *Green's Functions and Boundary Value Problems,* New York: Wiley, 1979.
5. R. F. Harrington, *Field Computation by Moment Methods,* New York: Macmillan, 1968.
6. T. K. Sarkar, A note on the choice of weighting functions in the method of moments, *IEEE Trans. Antennas Propag.,* **33**: 436–441, 1985.
7. T. K. Sarkar, A. R. Djordjevic, and E. Arvas, On the choice of expansion and weighting functions in the method of moments, *IEEE Trans. Antennas Propag.,* **33**: 988–996, 1985.
8. A. R. Djordjevic and T. K. Sarkar, A theorem on the moment methods, *IEEE Trans. Antennas Propag.,* **35**: 353–355, 1987.
9. J. M. Bornholdt and L. N. Medgyesi-Mitschang, Mixed domain Galerkin expansions in scattering problems, *IEEE Trans. Antennas Propag.,* **36**: 216–227, 1988.
10. R. W. P. King, *The Theory of Linear Antennas,* Cambridge, MA: Harvard Univ. Press, 1956.
11. D. S. Jones, *Methods in Electromagnetic Wave Propagation,* Oxford: Clarendon, 1979.
12. D. R. Wilton, S. M. Rao, A. W. Glisson, D. H. Schaubert, O. M. Al-Bundak, and C. M. Butler, Potential integrals for uniform and linear source distributions on polygonal and polyhedral domains, *IEEE Trans. Antennas Propag.,* **32**: 276–281, 1984.
13. A. W. Glisson and D. R. Wilton, Simple and efficient numerical methods for problems of electromagnetic radiation and scattering from surfaces, *IEEE Trans. Antennas Propag.,* **28**: 593–603, 1980.
14. S. M. Rao, A. W. Glisson, D. R. Wilton, and B. S. Vidula, A simple numerical solution procedure for statics problems involving arbitrary shaped surfaces, *IEEE Trans. Antennas Propag.,* **27**: 604–608, 1979.
15. S. M. Rao, D. R. Wilton, and A. W. Glisson, Electromagnetic scattering by surfaces of arbitrary shape, *IEEE Trans. Antennas Propag.,* **30**: 409–418, 1982.
16. S. M. Rao and P. K. Raju, Application of the method of moments to acoustic scattering from multiple bodies of arbitrary shape, *J. Acous. Soc. Amer.,* **86**: 1143–1148, 1989.
17. P. K. Raju, S. M. Rao, and S. P. Sun, Application of the method of moments to acoustic scattering from multiple infinitely long fluid filled cylinders, *Comp. Struct.,* **39**: 129–134, 1991.
18. S. M. Rao and B. S. Sridhara, Application of the method of moments to acoustic scattering from arbitrary shaped rigid bodies coated with lossless, shearless materials of arbitrary thickness, *J. Acous. Soc. Amer.,* **90**: 1601–1607, 1991.
19. S. M. Rao and B. S. Sridhara, Acoustic scattering from arbitrarily shaped multiple bodies in half space: Method of moments solution, *J. Acous. Soc. Amer.,* **91**: 652–657, 1992.
20. C. L. Bennett, *A Technique for Computing Approximate Electromagnetic Impulse Response of Conducting Bodies,* Ph.D. thesis, Purdue University, Lafayette, IN, 1968.
21. S. M. Rao, T. K. Sarkar, and S. A. Dianat, The application of the conjugate gradient method to the solution of transient electromagnetic scattering from thin wires, *Radio Sci.,* **19**: 1319–1326, 1984.
22. S. M. Rao, T. K. Sarkar, and S. A. Dianat, A novel technique to the solution of transient electromagnetic scattering from thin wires, *IEEE Trans. Antennas Propag.,* **34**: 630–634, 1986.
23. S. M. Rao and D. R. Wilton, Transient scattering by conducting surfaces of arbitrary shape, *IEEE Trans. Antennas Propag.,* **39**: 56–61, 1991.
24. D. A. Vechinski and S. M. Rao, Transient scattering from dielectric cylinders—E-field, H-field, and combined field solutions, *Radio Sci.,* **27**: 611–622, 1992.
25. D. A. Vechinski and S. M. Rao, Transient scattering from two-dimensional dielectric cylinders of arbitrary shape, *IEEE Trans. Antennas Propag.,* **40**: 1054–1060, 1992.
26. D. A. Vechinski and S. M. Rao, Transient scattering by conducting cylinders—TE case, *IEEE Trans. Antennas Propag.,* **40**: 1103–1106, 1992.
27. D. A. Vechinski and S. M. Rao, A stable procedure to calculate the transient scattering by conducting surfaces of arbitrary shape, *IEEE Trans. Antennas Propag.,* **40**: 661–665, 1992.
28. D. A. Vechinski, S. M. Rao, and T. K. Sarkar, Transient scattering from three-dimensional arbitrarily shaped dielectric bodies, *J. Optical Soc. Amer.,* **11**: 1458–1470, 1994.
29. E. K. Miller, L. Medgyesi-Mitschang, and E. H. Newman, *Computational Electromagnetics—Frequency-Domain Method of Moments,* New York: IEEE Press, 1992.
30. B. Z. Steinberg and Y. Leviatan, On the use of wavelet expansions in the method of moments, *IEEE Trans. Antennas Propag.,* **41**: 610–619, 1993.
31. R. Coifman, V. Rokhlin, and S. Wandzura, The fast multipole method: A pedestrian prescription, *IEEE AP-S Mag.,* **35**: 7–12, 1993.
32. F. X. Canning, Improved impedance matrix localization method, *IEEE Trans. Antennas Propag.,* **41**: 659–667, 1993.
33. M. Hestenes and E. Stiefel, Method of conjugate gradients for solving linear systems, *J. Res. Nat. Bur. Standards,* **49**: 409–436, 1952.

34. Y. Saad and M. H. Schultz, GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems, *SIAM J. Sci. Stat. Comp.,* **7**: 856–869, 1986.

S. M. Rao
G. K. Gothard
Auburn University

INTEGRAL EQUATIONS. See Integro-differential equations; Wavelet methods for solving integral and differential equations.

INTEGRAL TRANSFORMS. See Hankel transforms; Laplace transforms.

INTEGRATED ACOUSTOOPTIC DEVICES. See Acoustooptical devices.

INTEGRATED CIRCUIT MANUFACTURING DIAGNOSIS. See Diagnosis of semiconductor processes.