LAPLACE TRANSFORMS

In this article, we describe the fundamentals of the Laplace transform. This is one of the many integral transforms and is used primarily to simplify and solve differential equations, integral equations, and interconnected linear systems, among others. It is especially useful when convolution is involved. This occurs in both linear system theory and probability.

The Laplace Transform was introduced by Marquis Pierre-Simon de Laplace (1749–1827), the great French mathematician and astronomer. Many of the modern applications were developed by Oliver Heaviside (1850–1925), a British engineer. Heaviside was prolific and highly original, but he had many critics.

The Laplace transform is equal in importance to the Fourier transform, and both are mainstays of all undergraduate curricula in mathematics, physics, and engineering. They are also important in probability theory. Many of the examples in this article are drawn from electrical engineering.

The Laplace and Fourier transforms are the only integral transforms discussed here, but they are hardly the only integral transforms. Many of these are closely related to the Laplace transform and include the Mellin transform.

THE LAPLACE TRANSFORM AND LINEAR TIME INVARIANT SYSTEMS

Many systems in electrical engineering are linear and time invariant (LTI). Let L be a continuous time (CT) LTI system with input $x(t)$ and output $y(t) = L[x(t)]$. Let $\delta(t)$ be the Dirac delta, or CT impulse. Then $h(t) = L[\delta(t)]$, the response of the linear system to a unit impulse, is called the *impulse response.* Time invariance leads to the *convolution* representation of the system,

$$y(t) = (h * x)(t) = (x * h)(t) \tag{1}$$

$$= \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)\,d\tau \tag{2}$$

The limits of integration depend on the causality properties of both the input signal x and the impulse response h. If the system is causal, $h(t) = 0$, $\forall t < 0$. If the input also begins at time $t = 0$, then the integral becomes

$$y(t) = \int_0^t h(\tau)x(t-\tau)\,d\tau \tag{3}$$

An eigenvalue of the linear system L is a complex number $H = H(s)$, depending on a complex parameter $s = \sigma + j\omega$, that is associated with an eigenfunction $\phi(t; s)$, which satisfies

$$L[\phi(t;\, s)] = H(s)\phi(t;\, s) \tag{4}$$

The solution is

$$\phi(t;\, s) = \exp(st) \tag{5}$$

$$H(s) = \int_{-\infty}^{+\infty} e^{-st}h(t)\,dt \tag{6}$$

$H(s)$ is called the *transfer function* of the system and is the Laplace transform of $h(t)$.

The limits of integration extend only over the support of h or the set of t for which h is nonzero. If the limits are $0 \le t < \infty$, we work with the unilateral or one-sided Laplace transform. When the limits are $-\infty < t < +\infty$, this Laplace transform is called bilateral or two-sided. For any time function $x(t)$, we write $x \bullet\!\!\longrightarrow X$ or $X(s) = \mathscr{L}\{x(t)\}$ (equivalently $x(t) = \mathscr{L}^{-1}\{X(s)\}$) to denote the corresponding Laplace transform pair

$$X(s) = \textstyle\int x(t)e^{-st}\,dt$$

When the limits are unspecified, they are determined by the support of $x(t)$.

The integral defining the Laplace transform may exist only for some complex s, and this set is called the Region of Covergence (ROC) of the Laplace Transform for the time function $x(t)$. For the *unilateral* Laplace transform, the ROC is nonempty, and the Laplace transform exists (converges) for some s if and only if x is of exponential order

$$x(t) = O(e^{kt}) \text{ as } t \to \infty \tag{7}$$

where it is important to note that k can be positive.

In general, we say that f is "big-Oh of g,"

$$f(t) = O(g(t)) \text{ as } t \to \infty \text{ if } \lim_{t\to\infty} \frac{f(t)}{g(t)} < \infty$$

Then, f and g are said to be of the same order. For example, $\sin(t) = O(1)$.

Example 1. Exponentially Decaying Impulse Response. Here we consider the exponential impulse response

$$h(t) = Ae^{-at}u(t) \tag{8}$$

where $u(t)$ is the *unit step* function

$$u(t) := \int_0^t \delta(\tau)\,d\tau \tag{9}$$

$$= 1,\ t \ge 0 \tag{10}$$

$$= 0,\ t < 0 \tag{11}$$

and a is real. We find that

$$H(s) = \mathscr{L}\{h(t)\} \tag{12}$$

$$= \int_0^\infty Ae^{-at}e^{-st}\,dt \tag{13}$$

$$= \frac{A}{s+a}, \quad \text{ROC} = \{\mathscr{R}(s) > -a\} \tag{14}$$

where $\mathscr{R}$ denotes "real part of." This H is a *rational function,* a ratio of polynomials in s. Rational Laplace transforms arise frequently in the analysis of engineering systems and correspond to lumped element systems. The roots of the numerator polynomial are called the *zeros,* whereas the roots of the denominator are called the *poles.* This system has one pole at $s = -a$, and no finite zeros, although $|H| \to 0$ as $s \to \infty$.

The linear system associated with this exponential impulse response is called a *first-order* or *one-pole* system. The differential equation is

$$u'(t) + ay(t) = x(t) \tag{15}$$

where $y'(t) := dy(t)/dt$. It is interesting to note that the transfer function H can be determined directly from the differential equation through the eigenvalue property, without the intermediate step of finding the impulse response $h(t)$. Substituting $x(t) = \exp(-st)$ and $y(t) = H(s)x(t)$ into Eq. (15) we directly solve for $H(s) = A(s + a)^{-1}$.

We now allow the decay rate a to be complex, $a = b + j\beta$. The pole and ROC remain unchanged. The time function

$$x(t) = e^{-bt}\cos(\beta t)u(t)$$

can be simply obtained from earlier results. Treat s as *real-valued* for the moment, and use the linearity of the Laplace transform integral to see that

$$X(s) = \mathscr{L}\{\Re^{-(b+j\beta)t}\} \tag{16}$$

$$= \Re(s + b + j\beta)^{-1} \tag{17}$$

$$= \frac{s+b}{(s+b)^2 + \beta^2} \tag{18}$$

Finally, let s extend over the complete ROC, the half plane $\{\mathscr{R}(s) > -b\}$.

When $\mathscr{R}(a) < 0$, the ROC includes the $s = 2\pi jf$ axis. In this case, we say that the system L has a *frequency response* or Fourier transform. We will explore some basic properties via $\mathscr{H}(f) = H(s = 2\pi jf)$,

$$\mathscr{H}(f) = \int_{-\infty}^{+\infty} e^{-2\pi jft}h(t)\,dt \tag{19}$$

The input to the system is a pure complex tone, and oscillates without decay (Re$(s) = 0$). Revisiting the eigenvalue property, we confirm the fact that a monochromatic (pure tone) put into a linear system results in an output at the same frequency, with a shift in amplitude and phase.

Decompose $\mathscr{H}(f) = A(f)\exp[j\Phi(f)]$, into the magnitude response $A(f)$ and the phase response $\Phi(f)$ so that

$$A(f) = |\mathscr{H}(f)|, \quad \text{an even function} \tag{20}$$

$$\Phi(f) = \arg \mathscr{H}(f), \quad \text{an odd function} \tag{21}$$

For the one-pole system, we see that the Fourier Transform $\mathscr{H}$ is given by

$$\mathscr{H}(f) = \frac{A}{a + 2\pi jf} \tag{22}$$

$$\text{with } A(f) = \frac{A}{\sqrt{a^2 + (2\pi f)^2}} \tag{23}$$

$$\text{and } \Phi(f) = \tan^{-1}(2\pi f/a) \tag{24}$$

The Fourier and Laplace Transform is discussed in many undergraduate texts in engineering. A good starting point is is either of Papoulis' texts (1,2).

FIRST APPLICATIONS

Example 2. Linear Differential Equation with Constant Coefficients. The Laplace transform is especially useful in solving initial value problems. Consider

$$y''(t) + 3y(t) + 2y(t) = x(t) = e^{-3t}u(t) \text{ for } t \geq 0 \tag{25}$$

We seek a solution subject to the initial conditions

$$y'(0) = y_0' = -3 \tag{26}$$

$$y(0) = y_0 = 1 \tag{27}$$

Transforming both sides of Eq. (25) yields

$$[s^2Y(s) - sy_0 - y_0'] + 3[sY(s) - y_0] + 2Y(s) = \frac{1}{s+3} \tag{28}$$

Substituting $y_0 = +1$, $y_0' = -3$, we find

$$[s^2 + 3s + 2]Y(s) = s + X(s) = s + \frac{1}{s+3} \tag{29}$$

This can be interpreted as the superposition of response to the input $x(t)$ and the initial conditions. Define the transfer function

$$H(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)} \tag{30}$$

so that

$$Y(s) = H(s)X(s) + H(s)[sy_0 + y_0' + 3y_0] \tag{31}$$

Substituting and solving gives

$$Y(s) = \frac{s^2 + 3s + 1}{(s+1)(s+20)(s+3)} \tag{32}$$

A partial fraction expansion quickly yields

$$Y(s) = -\frac{1}{2}\frac{1}{s+1} + \frac{1}{s+2} + \frac{1}{2}\frac{1}{s+3} \tag{33}$$

The ROC is $\{\mathscr{R}(s) > -1\}$, so the inverse transform can be written

$$y(t) = [e^{-2t} + \frac{1}{2}(e^{-3t} - e^{-t})]u(t) \tag{34}$$

The slowest mode decays as $O(e^{-t})$, and corresponds to the largest pole at $s = -1$.

Example 3. Linear Feedback Control Systems Analysis. The Laplace transform is widely used to analyze the stability and response of linear feedback control systems. Consider the block diagram in Fig. 1.

Operating in steady state, we find that

$$Y(s) = K(s)E(s) \tag{35}$$

$$E(s) = X(s) - G(s)Y(s) \tag{36}$$

or

$$Y(s) = H(s)X(s) \tag{37}$$

where

$$H(s) = \frac{K(s)}{1 + K(s)G(s)} \tag{38}$$

K is called the *open-loop* transfer function, whereas H is the *closed-loop* transfer function. To illustrate the advantages of working in the s domain, we compare it to the time domain formulation. Consider the particular example of

$$K(s) = K/s \tag{39}$$

$$G(s) = s + \frac{1}{s + a} \tag{40}$$

so that

$$H(s) = \frac{K(s + a)}{K_1 s^2 + K_1 a s + K} \tag{41}$$

where $K_1 = K + 1$. The time domain system equations are

$$y(t) = K \int_{-\infty}^{t} e(\tau)\, d\tau \tag{42}$$

where

$$e(t) = x(t) - y(t) - \int_{-\infty}^{t} e^{-a\tau} y(t - \tau)\, d\tau \tag{43}$$

This integrodifferential equation is difficult to analyze or solve for particular inputs $x(t)$. However, when $X(s)$ is available, the solution is straightforward, given that we can invert $Y(s)$ in Eq. (37), with $H(s)$ given by Eq. (41).

Example 4. Solving Partial Differential Equations by the Laplace Transform. Laplace transforms can also be used to solve linear partial differential equations (PDEs). Examples include the diffusion equation and the wave equation. In this example we solve the diffusion equation, a linear PDE that is first order in time and second order in space. The problem prescribes boundary conditions in space and an initial condition in time.

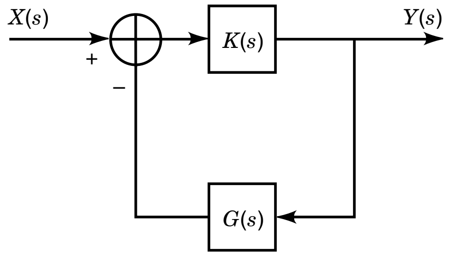

Figure 1. Control system block diagram.

Let $u = u(x, t)$ be twice continuously differentiable in x and t on $\{t \geq 0\}$ and $\{0 \leq x \leq L\}$. Denote partial derivatives by subscripts so that

$$u_{xx} = \frac{\partial^2}{\partial x^2} u(x, t) \quad u_t = \frac{\partial}{\partial t} u(x, t)$$

The diffusion equation with diffusion constant κ is written

$$u_{xx} = \frac{1}{\kappa} u_t, \quad t \geq 0,\ 0 \leq x \leq L \tag{44}$$

We assume that the boundary conditions are

$$u(0, t) = T_0 \tag{45}$$

$$u(x, 0) = 0 \tag{46}$$

Our approach is to first transform with respect to time t, and obtain an ordinary differential equation (ODE) for $U(x, s)$. Next, we solve for $U(x, s)$, subject to the boundary conditions. Finally, we invert to obtain $u = u(x, t)$.

Define

$$U(x, s) = \int_0^{\infty} u(x, t) e^{-st}\, dt \tag{47}$$

subject to

$$U(0, s) = \int_0^{\infty} u(0, t) e^{-st}\, dt \tag{48}$$

$$= T_0/s \tag{49}$$

Using basic properties of the Laplace transform, we reduce the PDE to an ODE in x. Transforming gives

$$U_{xx}(x, s) - \frac{1}{\kappa}[sU(x, s) - u(x, 0)] = 0 \tag{50}$$

Substituting the boundary condition gives the ODE, which we view as parametric in s,

$$I_{xx}(x, s) = \frac{s}{\kappa} U(x, s) \tag{51}$$

subject to

$$U(0, s) = \frac{T_0}{s} \tag{52}$$

Solving this for a fixed s yields

$$U(x, s) = \frac{T_0}{s} \exp\left(-x\sqrt{\frac{s}{\kappa}}\right) \tag{53}$$

The remaining task is to carry out the inversion and find

$$u(x, t) = \mathscr{L}^{-1}\left\{\frac{T_0}{s} \exp\left(-x\sqrt{\frac{s}{\kappa}}\right)\right\} \tag{54}$$

We will solve this inversion problem using a roundabout approach. Differentiate with respect to x, and write out the inversion integral to obtain

$$u_x(x, t) = \int \frac{ds}{2\pi j} \left(-\frac{T_0}{\sqrt{s\kappa}}\right) e^{-x\sqrt{\frac{s}{\kappa}}} e^{st}$$

Substitute $\sqrt{s} = j\omega$ and simplify to obtain

$$u_x(x, t) = \frac{-2T_0}{\sqrt{\kappa}} \int \frac{d\omega}{2\pi} e^{-j\omega\frac{x}{\sqrt{\kappa}} - t\omega^2}$$

Completing the square in ω,

$$-t\omega^2 - j\omega(x/\sqrt{\kappa}) = -t\left(\omega - \frac{jx}{2t\sqrt{\kappa}}\right)^2 - \frac{x^2}{4\kappa t}$$

we substitute and simplify to get

$$u_x(x, t) = \frac{-2T_0}{\sqrt{\kappa}} e^{-\frac{x^2}{4\kappa t}} \int \frac{d\omega}{2\pi} \exp\left\{-t\left(\omega - \frac{jx}{2t\sqrt{\kappa}}\right)^2\right\}$$

This last integral is evaluated by normalization because it is essentially the probability mass under the Gaussian curve, and we find that

$$u_x(x, t) = \frac{-T_0}{\sqrt{\pi\kappa t}} e^{-x^2/4\kappa t} \quad (55)$$

Integrating with respect to x yields

$$u(x, t) = T_0\left[1 - \text{erf}\left(\frac{x}{2}\sqrt{\frac{1}{\kappa t}}\right)\right] \quad (56)$$

where $\text{erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-t^2}\, dt$.

BASIC PROPERTIES OF UNILATERAL LAPLACE TRANSFORMS

In this section, we list a number of the basic properties of the unilateral or "one-sided" Laplace transform, adopted from Henrici (3). We let $f(t)$ denote a time function, and $F(s)$, the corresponding transform, $f \bullet\!\!\longrightarrow F$. We assume that $f(t)$ satisfies some reasonable properties.

1. $f(t)$ is identically zero for all negative time

$$f(t) = 0,\ \forall t < 0 \quad (57)$$

2. $f(t)$ is continuous except for a countable number of step discontinuities on the nonaccumulating set of points $0 < t_1 < t_2 < \ldots$
3. $f(t)$ is absolutely integrable ($f \in L_1[0, \infty]$)

$$\int_0^\infty |f(t)|\, dt < \infty \quad (58)$$

Here the Laplace transform of $f(t)$, denoted $f(t) \bullet\!\!\longrightarrow F(s)$ or $F(s) = \mathscr{L}\{f(t)\}$, consists of *both* the complex function

$$F(s) = \int_0^\infty f(t)e^{-st}\, dt \quad (59)$$

and the associated ROC, $\mathscr{R}(s) > s_0$. Many of the properties continue to hold for suitably defined time functions on the whole real line $-\infty < t < +\infty$.

Property 1. Inversion and Uniqueness. A Laplace transform $F(s)$ and its ROC can be inverted to a unique time function $f(t)$. Note that not all functions of a complex variable s are Laplace transforms. Techniques to carry out the inversion are discussed later. Usually inversion is the most difficult part of the process, either theoretically or computationally.

Property 2. Linearity. If $f \bullet\!\!\longrightarrow F$ and $g \bullet\!\!\longrightarrow G$, then for constants a, b,

$$af + bg \bullet\!\!\longrightarrow aF + bG \quad (60)$$

This extends directly for finite sums.

Property 3. Analyticity. Within its ROC, $F(s)$ is an analytic function. This implies that derivatives of all orders exist and can be computed by

$$\frac{d^k}{ds^k} F(s) = \int_0^\infty (-t)^k e^{-st}\, dt \quad (61)$$

Property 4. Decay in $F(s)$.

$$\lim_{s\to\infty} F(s) = 0 \quad (62)$$

where the limit is taken along any ray lying within the ROC of F.

Property 5. Shifting.

$$f(t - t_0) \bullet\!\!\longrightarrow e^{-st_0} F(s) \quad (63)$$

Property 6. Scaling.

$$f(at) \bullet\!\!\longrightarrow \frac{1}{a} F\left(\frac{s}{a}\right) \quad (64)$$

Property 7. Differentiation. When $f'(t) \in L_1$ with an initial value $f(0+)$,

$$f'(t) \bullet\!\!\longrightarrow sF(s) - f(0+) \quad (65)$$

When $f(t)$ is sufficiently differentiable and $f^{(n)}(t) := (d/dt)^n f(t)$ with $f^{(0)}(t) := f(t)$, we find that

$$f^{(n)}(t) \bullet\!\!\longrightarrow s^n F(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^{(k)}(0+) \quad (66)$$

For example,

$$f''(t) \bullet\!\!\longrightarrow s^2 F(s) - sf(0+) - f'(0+) \quad (67)$$

This is the most useful property in solving initial value problems that arise from linear circuits or mechanical problems.

Property 8. Integration.

$$\int_0^t f(\tau)\, d\tau \bullet\!\!\longrightarrow \frac{1}{s} F(s) \quad (68)$$

Property 9. Convolution. If $f(t) \bullet\!\!\longrightarrow F(s)$ and $h(t) \bullet\!\!\longrightarrow H(s)$, then

$$(h * f)(t) = \int_0^t h(\tau) f(t-\tau)\, d\tau \bullet\!\!\longrightarrow F(s)H(s) \tag{69}$$

Property 10. Multiplication. If $f(t) \bullet\!\!\longrightarrow F(s)$ and $g(t) \bullet\!\!\longrightarrow G(s)$, then

$$f(t)g(t) \bullet\!\!\longrightarrow \int_C \frac{d\lambda}{2\pi j} G(\lambda)H(s-\lambda) \tag{70}$$

An important special case is when $s = 0$, and yields a Parseval theorem for the Laplace transform

$$\int_0^\infty f(t)g(t)\, dt = \int_C \frac{d\lambda}{2\pi j} F(\lambda)G(-\lambda) \tag{71}$$

In both cases, a suitable contour of integration in the complex λ plane is required. This contour C is vertical and lies within the ROC of both F and G, $\mathrm{ROC}_F \cap \mathrm{ROC}_G$. If this intersection is empty, the resulting integral does not exist.

Property 11. Periodic Functions. Let $f_0(t)$ be defined over the fundamental period $0 < t < T$, and let

$$f(t) = \mathrm{rep}_T[f_0(t)] := \sum_{n=-\infty}^{\infty} f_0(t+nT) \tag{72}$$

so that $f(t) = f(t+T)$ is T periodic. Then if $f_0(t) \bullet\!\!\longrightarrow F_0(s)$,

$$f(t) \bullet\!\!\longrightarrow F(s) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt \tag{73}$$

Property 12. Multiplication by t^k.

$$t^k f(t) \bullet\!\!\longrightarrow (-1)^k F^{(k)}(s) \tag{74}$$

This property is very useful in quickly evaluating Laplace transforms. For example, from Example 1 we know that

$$e^{-t} \bullet\!\!\longrightarrow (s+1)^{-1} \tag{75}$$

Using this property, we immediately see that

$$\frac{t^{k-1}e^{-t}}{k!} \bullet\!\!\longrightarrow (s+1)^{-k} \tag{76}$$

ADVANCED TRANSFORM PAIRS

In this section we present further properties, and applications, of a more advanced nature.

Log Functions

An Application of Differentiation. Assume that a time function, $f = f(t;\, \nu)$, depends on a parameter ν. We further assume that f is analytic in ν on some open set $\mathscr{D}$, so that derivatives of any order k, $(d/d\nu)^k f(t,\, \nu)$, exist. Then if

$$F(s;\, \nu) = \mathscr{L}\{f(t;\, \nu)\} \tag{77}$$

we find that

$$(d/d\nu)^k F(s;\, \nu) = \mathscr{L}\{(d/d\nu)^k f(t;\, \nu)\} \tag{78}$$

We apply this result to the determination of $\mathscr{L}\{\log t\}$.

Recall that the Laplace transform of t^ν can be found from $\Gamma(x)$, the Gamma function,

$$\Gamma(\nu) = \int_0^\infty t^{\nu-1} e^{-t}\, dt \tag{79}$$

The correspondence is

$$\mathscr{L}\{t^\nu\} = \int_0^\infty t^{\nu-1} e^{-st}\, dt \tag{80}$$

$$= s^{-\nu-1}\Gamma(\nu+1) \tag{81}$$

Note that both sides are analytic functions of ν on the ν plane, cut on the negative real ν axis. Differentiate with respect to ν, and evaluate the result at $\nu = 0$ to find

$$\mathscr{L}\{\log t\} = -\frac{1}{s}(\log s + \gamma) \tag{82}$$

where $\gamma := -\Gamma'(1) = 0.57721566$ is Euler's constant. The method can be repeated, for example, to find $\mathscr{L}\{\log^2 t\}$.

An Application of Integration. Again consider the time function, $f = f(t;\, \nu)$, depending on a parameter ν. We further assume that f is integrable with respect to ν, so that $\int f(t;\, \nu)\, d\nu$ exists. Then, if $F(s;\, \nu) = \mathscr{L}\{f(t;\, \nu)\}$,

$$\mathscr{L}\{\textstyle\int f(t;\, \nu)\, d\nu\} = \textstyle\int F(s;\, \nu)\, d\nu \tag{83}$$

As an example, recall that

$$\mathscr{L}\{\cos(bt)\} = \frac{s}{s^2+b^2} \tag{84}$$

Integrating both sides with respect to b yields the transform pair

$$\mathscr{L}\left\{\frac{\sin(bt)}{t}\right\} = \tan^{-1}(b/s) \tag{85}$$

Series Expansions

Hardy's theorem is frequently invoked to find the transform or inverse transform of a time function given as a convergent series. There are two closely related variants: for Laplace transforms and for inversions.

Let

$$f(t) = t^\nu \sum_{n=0}^{\infty} a_n t^n \tag{86}$$

and consider $F(s)$ obtained by term-by-term integration of f,

$$\mathscr{L}\{f(t)\} = \sum_{n=0}^{\infty} a_n \int_0^{\infty} t^{\nu+n} e^{-st}\, dt \tag{87}$$

$$F(s) = \sum_{n=0}^{\infty} a_n \Gamma(\nu+n+1) s^{-\nu-n-1} \tag{88}$$

Hardy's Theorem. If $F(s)$ is convergent for some $s = s_0 > 0$, then $f(t)$ converges for all $t > 0$ and $F(s) = \mathscr{L}\{f(t)\}$.

The second version is a converse.

Corollary. Let

$$F(s) = s^{-\nu} \sum_{n=0}^{\infty} c_n s^{-n} \tag{89}$$

which converges for some $|s| \geq \rho > 0$ and $|\arg s| < \pi$. Then $F(s)$ is the analytic continuation of $\mathscr{L}\{f(t)\}$, where

$$f(t) = \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(\nu+n)} t^{\nu+n-1} \tag{90}$$

This provides a useful inversion theorem.

We apply this to find the Laplace transform for the Bessel function J_0. Consider

$$F(s) = s^{-1} e^{-a/s} \tag{91}$$

$$= s^{-1} \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} s^{-n} \tag{92}$$

Applying the corollary and inverting term by term gives

$$\mathscr{L}^{-1}\{F(s)\} = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} s^{-n} \tag{93}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{at})^{2n}}{n!n!} \tag{94}$$

$$= J_0(2\sqrt{at}) \tag{95}$$

The Saddlepoint Approximation and Numerical Contour Integration

Many times the inversion integral cannot be carried out analytically. This often occurs when the integrand contains branch cuts and essential singularities (2), or when the number of poles is so large as to preclude numerical summation of the residue series. In these cases, techniques from asymptotic analysis (3–5) suggest some useful numerical methods. We refer to these as *saddlepoint* methods.

Consider the inversion integral

$$f(t) = \int_C \frac{ds}{2\pi j} F(s) e^{st}$$

where C is a suitable contour. We will consider the generic form

$$f(t) = \int_C \frac{ds}{2\pi j} e^{\phi(s)}$$

where

$$\phi(s) = st + \log F(s)$$

The method relocates the path of integration C so that it passes through a saddlepoint s_0, where

$$\phi'(s_0) = \frac{d}{ds}\phi(s)|_{s_0} = 0$$

In many cases, especially for large t, the main contribution to $f(t)$ arises from points on C close to s_0. Expand $\phi(s)$ about the point s_0,

$$\phi(s) = \phi(s_0) + \phi''(s_0)\frac{(s-s_0)^2}{2!} + \cdots$$

because $\phi'(s_0) = 0$. Substituting, with $s = s_0 + jy$, we find

$$f(t) = \frac{e^{\phi_0}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{ds}{\sqrt{s\pi}} e^{-\frac{\phi_0'' y^2}{2}} \tag{96}$$

where $\phi_0 = \phi(s_0)$, $\phi_0'' = \phi''(s_0)$. The integral is evaluated by normalization to give the saddlepoint approximation

$$f(t) = \frac{e^{\phi_0}}{\sqrt{2\pi\phi_0''}} \tag{97}$$

The methods can be extended by numerically integrating along the "steepest descent" contour from the saddlepoint. For applications to probability, and in particular in communications and statistical signal processing, see Refs. 6 to 8.

APPLICATIONS OF THE LAPLACE TRANSFORM IN PROBABILITY

The Laplace Transform is a natural tool for handling many distributional problems in applied probability and mathematical statistics, especially those involving linear combinations of statistically independent positive random variables. For these, a natural integral transform is the one-sided Laplace transform. We begin this section with some basic concepts about random variables and their distributions and lead into the use of transform techniques. Appropriate references include Papoulis (9) and Feller (10). Because the inversion problem is usually the most difficult step in the analysis, some approximate methods from asymptotic analysis are discussed. Detection theory at the level of Helstrom (6) is a source of many of our applications.

Random Variables and Their Distributions

Let $\mathbf{x}$ denote a positive random variable (rv) with probability density function (pdf) $f(x)$, distribution function (df) $F(x)$, and exceedance $\overline{F}(x) := 1 - F(x)$. The pdf is a positive integrable function defined on $(0, \infty)$, with unit normalization,

$$\int_0^{\infty} f(x)\, dx = 1$$

The probability that the rv $\mathbf{x}$ falls into some interval $[a, b]$ where $0 \le a \le \infty$ can be determined from either f or F,

$$\Pr\{a < \mathbf{x} \le b\} = F(b) - F(a) \quad (98)$$

$$= \int_a^b f(x)\,dx \quad (99)$$

Do not confuse the df $F(x)$ with a Laplace transform. The pdf is also used in the concept of the *expectation* of a measurable function of a rv. For a suitable g, define

$$E\{g(x)\} = \int_0^\infty g(x) f(x)\,dx$$

The *moment generating function* (mgf) is defined as the Laplace transform of the density $f(x)$. Alternatively, it can be interpreted as an expectation. We will use $h = h(s)$ or $h_{\mathbf{x}}(s)$ to denote

$$h(s) = E\{\exp(-\mathbf{x}s)\} \quad (100)$$

$$= \int_0^\infty e^{-xs} f(x)\,dx \quad (101)$$

Note the distinction between the rv $\mathbf{x}$ and the parameter x. It is this probabilistic interpretation of the mgf which makes it so useful in theory and application. The mgf exists within the ROC of the Laplace transform. By normalization of the pdf, $h(s = 0) = 1$. In fact, other properties hold. Bernstein's theorem states that a function $h(s)$ is a mgf if and only if it is a *completely monotonic* (cm) function

$$(-1)^k \frac{d^k}{ds^k} h(s) \ge 0 \; \forall s \ge 0$$

In many of our applications in detection theory, *tail probabilities* are of interest. The right-hand tail is simply

$$\overline{F}(x) = \int_x^\infty f(x)\,dx$$

where x is so large that $\overline{F}(x) \ll 0.5$. For example, to find P_e, the probability of error of a digital communications system, we are often asked to evaluate tail probabilities of the order of 10^{-5} to 10^{-10}.

The distribution is a complete statistical description of the random variable $\mathbf{x}$. Often simpler descriptors suffice. The most common are the *moments* of the rv. Define μ_k, the kth moment of the rv $\mathbf{x}$, by

$$\mu_k = E\{\mathbf{x}^k\} = \int_0^\infty x^k f(x)\,dx \quad (102)$$

Moment generating function refers to the fact that the moments are determined by differentiating $h(s)$. Expanding e^{-xs} and integrating term by term gives

$$h(s) = \int_0^\infty e^{-xs} f(x)\,dx \quad (103)$$

$$= \sum_{k=0}^\infty \int \frac{(-x)^k}{k!} f(x)\,dx \quad (104)$$

$$= \sum_{k=0}^\infty (-1)^k \frac{\mu_k}{k!} \quad (105)$$

Equating this moment expansion with the Taylor series for $h(s)$ gives

$$\mu_k = E\{x^k\} = (-1)^k h^{(k)}(0)$$

where the kth derivative is denoted $h^{(k)}(s) := (d/ds)^k h(s)$. Thus, the moments can be easily determined from the mgf.

One of the most important uses of the mgf is in analyzing the distribution of sums of independent random variables. Let $\mathbf{t}_1, \ldots, \mathbf{t}_n$ be a collection of n independent and identically distributed (iid) component random variables with common pdf $p(x)$ and mgf $g(s) = E\{\exp(-\mathbf{t}s)\}$. Then if $\mathbf{x} = \mathbf{t}_1 + \cdots + \mathbf{t}_n$,

$$h(s) = E\{\exp(-\mathbf{x}s)\} \quad (106)$$

$$= E\left\{\exp\left(-s\sum_{j=1}^n \mathbf{t}_j\right)\right\} \quad (107)$$

$$= \prod_{j=1}^n E\{\exp(-s\mathbf{t}_j)\} \quad (108)$$

$$= g^n(s) \quad (109)$$

The mgf of the sum of n iid random variables is nth power of the mgf of the individual components. This extends easily to the nonidentically distributed case. We illustrate many of these general properties with an example.

Sums of Independent Exponentially Distributed Random Variables. Again let $\mathbf{t}_1, \ldots, \mathbf{t}_n$ be a collection of n iid component random variables with common pdf $p(x)$ and mgf $g(s) = E\{e^{-\mathbf{t}s}\}$. Of interest is the distribution of the sum, $\mathbf{x} = \mathbf{t}_1 + \cdots + \mathbf{t}_n$, under an exponential assumption on the distribution of the component random variables. When the $\mathbf{t}$ are exponentially distributed with mean $E\{\mathbf{t}\} = \mu$,

$$f(t) = \frac{1}{\mu} e^{-t/\mu}, \quad t \ge 0 \quad (110)$$

$$F(t) = 1 - e^{-t/\mu}, \quad t \ge 0 \quad (111)$$

$$g(s) = \frac{1}{1 + \mu s} \quad (112)$$

The moments are easily determined by differentiation,

$$\mu_k = E\{\mathbf{t}^k\} = k!\mu^k$$

The mgf of $\mathbf{x}$, the sum of the iid exponential components, is

$$h(s) = (1 + \mu s)^{-n}$$

To determine the density or distribution, we must invert this Laplace transform. The pdf is given by the contour integral representation of the inverse transform

$$f(x) = \int_{\mathrm{C}} \frac{ds}{2\pi j} h(s) e^{xs}$$

where C is a vertical contour in the complex s plane lying in the region of convergence. In our example, the ROC is the half place $\mathscr{R}(s) > -1/\mu$. The density can be obtained using the method of residues. Closing the contour in the left half plane, and using Cauchy's integral formula,

$$f(x) = \oint \frac{ds}{2\pi j} \frac{e^{xs}}{(1+\mu s)^n} \quad (113)$$

$$= \frac{1}{\mu} \left(\frac{x}{\mu}\right)^{n-1} \frac{e^{-x/\mu}}{(n-1)!}, \quad x > 0 \quad (114)$$

In many problems, the distribution F or exceedance $\overline{F}$ is of more interest than the density f. General contour integral representations for the distribution and exceedance is

$$\overline{F}(x) = -\int_{C-} \frac{ds}{2\pi j} s^{-1} h(s) e^{xs} \quad (115)$$

$$F(x) = \int_{C+} \frac{ds}{2\pi j} s^{-1} h(s) e^{xs} \quad (116)$$

The contours C+, C− are both vertical and lie in the ROC of mgf $h(s)$ as shown in Fig. 2. The contour C− crosses the *negative* real s axis, whereas the contour C+ crosses the *positive* real s axis.

To obtain the cdf of $\mathbf{x}$, we will compute $\overline{F}(x)$ and obtain F by subtraction, $F(x) = 1 - \overline{F}(x)$. From the contour integral, with $h(s) = (1 + \mu s)^{-n}$, closing the contour are the pole at $s = -1/\mu$, and invoking Cauchy's integral formula, we find that

$$\overline{F}(x) = \int_{C-} \frac{ds}{2\pi j} \frac{(-s)^{-1} e^{xs}}{(1+\mu s)^n} \quad (117)$$

$$= \frac{\mu^{-n}}{(n-1)!} \left(\frac{d}{ds}\right)^{n-1} [(-s)^{-1} e^{xs}]_{1/\mu} \quad (118)$$

To complete the derivation, we must carry out the differentiation using Leibnitz's rule,

$$\left(\frac{d}{ds}\right)^m h(s) g(s) = \sum_{k=0}^{m} \binom{m}{k} h^{(k)}(s) g^{(m-k)}$$

to obtain a residue series. Simplifying gives

$$\overline{F}(x) = \sum_{k=0}^{n-1} \frac{1}{\mu} \left(\frac{x}{\mu}\right)^k \frac{e^{-x/\mu}}{k!}$$

The derivation of this formula is not the end of the story. When $\overline{F}(x)$ is small, that is, for large n and $x \gg n\mu = E\{\mathbf{x}\}$, the sum contains many terms and is difficult to evaluate as a result of the disparity in magnitude between the large $(x/\mu)^k$ and the small $\exp(-x/\mu)$. This disparity causes overflow, which must be handled carefully. The point is that a residue series often leads to numerical instability when the number of poles, or their multiplicity, is large.

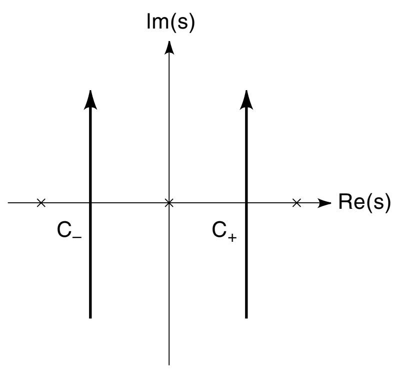

Figure 2. Bromwich contour definitions.

Upper bounds on the exceedance $\overline{F}$, which are tight and easy to compute, are also important. A general technique to obtain these is based on Markov's inequality.

Markov's Inequality. Here $\mathbf{x}$ has exceedance $\overline{F}(x)$ and mgf $h(s)$. If $v(x; s)$ is a positive nondecreasing function for $x \in [0, \infty]$ and with parameter s,

$$\overline{F}(x_0) \le \min_s E\{v(\mathbf{x}; s)/v(x_0; s)\}$$

The most important special cases include the Chernoff bound, where $v(x; s) = \exp(-sx)$, $s \le 0$, and the moment bound, where $v(x; s) = (x)^s$, $s \ge 0$. The Chernoff bound is

$$\overline{F}(x_0) \le \min_{s \le 0} \{e^{sx_0} h(s)\}$$

For the example, we find that

$$\overline{F}(x_0) \le \min_{s \le 0} \{e^{sx_0} (1+\mu s)^{-n}\}$$

The best choice of $s = s_0 \le 0$ is at $s_0 = (n/x_0) - (1/\mu)$, which is negative for $x > n\mu = E\{\mathbf{x}\}$. Substituting s_0 yields the Chernoff bound,

$$\overline{F}(x) \le \frac{(x/\mu)^n e^{-x/\mu}}{n^n e^{-n}}$$

The denominator is asymptotic to $n!/\sqrt{2\pi n}$. An important feature of the Chernoff bound is the exponential rate of decay which captures the dominant features of the exact exceedance. The Laplace transform clearly plays a prominent role in the bound. These bounds can be developed into approximations using saddlepoint methods.

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