LYAPUNOV METHODS

Phenomena that change are most suitably described in terms of their states evolving in time. These are called dynamical systems. Such systems, which occur in nature or are man-made, are frequently endowed with natural states, called equilibria (or rest positions) or operating points. When after a disturbance, the system states return to an equilibrium (operating point), one speaks of a stable equilibrium. In the case of man-made objects, a great deal of engineering is concerned with the design of systems with stable operating points (equilibria). The most universally accepted notion of stability is Lyapunov stability. In the present article we introduce this concept and we present results for analyzing the Lyapunov stability properties of an equilibrium. Collectively, such results are called Lyapunov Method.

For a given set of initial conditions, the characterization of a physical system is usually given in terms of the evolution in time of a motion in state space. If we let t_0 and $x(t_0) = x_0$ denote initial time and initial state, respectively, we can represent the motion by a function $p(\cdot, x_0, t_0)$ from R^+ to X, that is, $p(\cdot, x_0, t_0): [t_0, \infty) \to X$ where $[t_0, \infty) \subset R^+ = [0, \infty)$ denotes real time and X denotes the state space (some metric space). Now if we let initial state x_0 vary over a specified set A in X ($x_0 \in A \subset X$) and if we let initial time t_0 vary over a specified set R_0^+ in R^+ ($t_0 \in R_0^+ \subset R^+$), there will result a family of motions that we will denote by S, provided that $p(t_0, x_0, t_0) = x_0$. The resulting five-tuple $\{X, S, A, R^+, R_0^+\}$ is called a dynamical system. When there is no room for confusion, we will simply speak of a dynamical system S, rather than a dynamical system $\{X, S, A, R^+, R_0^+\}$.

In the above discussion, the evolution of motions is along real time t in R^+ (i.e., $t \in R^+$). In this case, we speak of continuous-time dynamical systems. In many applications, motions may also take place along discrete instants, for example, along the nonnegative integers, $Z^+ = \{0, 1, 2, 3, \cdots\}$, resulting in a discrete-time dynamical system that we denote by $\{X, S, A, Z^+, Z_0^+\}$, where Z_0^+ is in Z^+ (i.e., $Z_0^+ \subset Z^+$). Still, in other types of applications, some components of the motions may evolve along R^+, while others may evolve, e.g., along Z^+, so that the entire motion will evolve along a subset T of $R^+ \times Z^+$ (i.e., $T \subset R^+ \times Z^+$). The resulting dynamical systems $\{X, S, A, T, T_0\}$ are called hybrid dynamical systems.

Examples of continuous-time dynamical systems are systems whose motions are determined by the solutions of systems of ordinary differential equations and systems of ordinary differential inequalities while examples of discrete-time dynamical systems include systems whose motions are determined by the solutions of ordinary difference equations and systems of ordinary difference inequalities. All of these are examples of finite dimensional dynamical systems.

If a dynamical system is not finite dimensional, it is said to be infinite dimensional. Examples of infinite dimensional dynamical systems include those whose motions are determined by the solutions of delay differential equations, functional differential equations, partial differential equations, Volterra integrodifferential equations, and the like.

In addition to the above, dynamical systems may also be determined by "equation free" characterizations (discrete event systems, systems determined by Petri nets, and the like), and by mixtures of equations [hybrid dynamical systems, such as, digital control systems consisting of a continuous-time plant and a digital (discrete-time) controller].

Dynamical systems that represent processes that are either manufactured or can be found in nature are usually endowed with one or more "operating points." Mathematically, these are represented by *invariant sets.* A set M in A (i.e., $M \subset A \subset X$) is said to be an invariant set (with respect to S) if whenever a motion at t_0 starts out in M will remain in M forever (i.e., if $p(t_0, x_0, t_0) = x_0 \in M$, then $p(t, x_0, t_0) \in M$ for all $t \geq t_0 \geq 0$). If in particular, M consists of one single point, say x_e, then x_e is called an equilibrium of the dynamical system S. In this case $M = \{x_e\}$ and $p(t, x_e, t_0) = x_e$ for all $t \geq t_0$.

In the following, we will confine ourselves to equilibria. A discussion for general invariant sets would follow along similar lines, involving obvious modifications.

The qualitative behavior of motions of a dynamical system in the vicinity of an operating point (i.e., in the vicinity of an equilibrium) is of great interest in applications and gives rise to the various stability notions of an equilibrium in the sense of Lyapunov.

Suppose that x_e is an equilibrium of a dynamical system S. If by choosing all the initial points of the motions in a sufficiently small neighborhood of x_e, we can force the motions to stay sufficiently close to x_e for all $t \geq t_0 \geq 0$ (in terms of the metric of X), the equilibrium x_e is said to be stable (in the sense of Lyapunov). If x_e is stable, and if by choosing all initial points of the motions in some neighborhood of x_e at $t = t_0$, we can force the motions to tend to x_e as t becomes arbitrarily large (i.e., as $t \to \infty$), then x_e is said to be asymptotically stable (in the sense of Lyapunov). The set of initial points for which the above statement is true is called the domain of attraction of x_e. If the above statement is true for all initial points (i.e., for all motions), then x_e is said to be globally asymptotically stable. In this case, x_e is the only equilibrium of the dynamical system. If x_e is asymptotically stable and if "the motions tend to x_e exponentially" (with respect to the metric of X), then x_e is said to be exponentially stable. Finally, if x_e is not stable, it is said to be unstable.

Other closely related qualitative attributes of dynamical systems concern various notions of boundedness of motions. These comprise the Lagrange stability of dynamical systems.

In the qualitative analysis of dynamical systems, Lyapunov methods play a central role. The aim is to ascertain qualitative properties of families of motions near an equilibrium point (in the sense discussed above) without having to actually determine explicit expressions for the motions of a dynamical system. This is fortunate, for in general, there are no known techniques that yield explicit expressions for such motions. It is for this reason that one frequently speaks of the Direct Method of Lyapunov (of stability analysis). In addition to determining various stability properties of an equilibrium, the Direct Method of Lyapunov can also be used in determining various boundedness properties of motions of dynamical systems (Lagrange stability).

The Direct Method of Lyapunov employs auxiliary scalar-valued functions of the system state, called Lyapunov functions, which frequently are viewed as (generalized) energy functions for dynamical systems, or as (generalized) distance functions [from the motions (at time t) to an equilibrium] for dynamical systems. Stability properties of an equilibrium (or boundedness properties of motions) are then deduced from the behavior of the Lyapunov functions evaluated along the motions of a dynamical system. In general, this can be accomplished without explicitly determining expressions for the motions of a given dynamical system; hence, the term the Direct Method of Lyapunov.

To make the above discussion more precise, we will in the following confine ourselves to finite-dimensional, continuous-time dynamical systems whose motions are determined by systems of ordinary differential equations. In this case, the state space is given by $X = R^n$, the metric on X is determined by any one of the equivalent norms, $|\cdot|$, on R^n, and we will assume that $R_0^+ = R^+$.

DYNAMICAL SYSTEMS DETERMINED BY ORDINARY DIFFERENTIAL EQUATIONS

We shall concern ourselves with dynamical systems that are determined by the solutions of first-order ordinary differential equations of the form

$$\dot{x} = f(x, t) \tag{E}$$

where $x = (x_1, \cdots, x_n)^T \in R^n$ (i.e., x is a real n-vector), $t \in R^+ = [0, \infty)$ (i.e., $t \geq 0$), $\dot{x}$ denotes differentiation with respect to t (i.e., $\dot{x} = (\dot{x}_1, \cdots, \dot{x}_n)^T$, $\dot{x}_i = dx_i/dt$, $i = 1, \cdots, n$), and f is a continuous function of $R^n \times R^+$ into R^n (i.e., $f(x, t) = [f_1(x_1, \cdots, x_n, t), \cdots, f_n(x_1, \cdots x_n, t)]^T = [f_1(x, t), \cdots, f_n(x, t)]^T$ where it is assumed that $f_i(x, t)$ is continuous on $R^n \times R^+$, $i = 1, \cdots, n$). Unless otherwise stated, we assume that for every (x_0, t_0), the initial-value problem

$$\dot{x} = f(x, t),\ x(t_0) = x_0 \tag{I}$$

possesses a unique solution $p(t, x_0, t_0)$ with $p(t_0, x_0, t_0) = x_0$, which is defined for all $t \geq t_0$ and which depends continuously on the initial conditions (x_0, t_0).

A point x_e in R^n is called an equilibrium point of (E) if $f(x_e, t) = 0$ for all $t \geq 0$. Other terms for equilibrium point include stationary point, singular point, critical point, and rest position. We note that if x_e is an equilibrium point of (E), then for any $t_0 \geq 0$, $p(t, x_e, t_0) = x_e$ for all $t \geq t_0$ [i.e., x_e is a unique solution of (E) with initial conditions given by $p(t_0, x_e, t_0) = x_e$].

As a specific example, consider the simple pendulum that is described by equations of the form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k \sin x_1,\ k > 0 \end{aligned} \tag{1}$$

where x_1 denotes angular displacement and x_2 denotes angular velocity of a mass subjected to gravitational force and rotating about a fixed point. By convention, we let $x_1 = 0$ when the mass position is in the most downward position. Physically, the pendulum has two equilibrium points. One of these is when $x_1 = x_2 = 0$ (when the mass is in the most downward position) and the second point is when $x_1 = \pm\pi$ and $x_2 = 0$. However, the model of the pendulum, represented by system (1), has countably infinitely many equilibrium points which are located at the points $(\pi n, 0)$, $n = 0, \pm 1, \pm 2, \cdots$.

An equilibrium point x_e of (E) is called an isolated equilibrium point if there is an $r > 0$ such that $B(x_e, r) \subset R^n$ contains no equilibrium points of (E) other than x_e itself. Here, $B(x_e, r) = \{x \in R^n : |x - x_e| < r\}$, where $|\cdot|$ denotes any one of the equivalent norms on R^n. (Thus, $B(x_e, r)$ denotes a sphere in R^n with center at x_e and radius equal to $r > 0$.)

All equilibrium points of system (1) are isolated equilibria in R^2. On the other hand, for a dynamical system described by the system of equations

$$\begin{aligned} \dot{x}_1 &= -ax_1 + bx_1x_2 \\ \dot{x}_2 &= -bx_1x_2 \end{aligned} \tag{2}$$

where $a > 0$, $b > 0$ are constants, every point on the positive x_2-axis is an equilibrium point for system (2).

It should be noted that there are systems with no equilibrium points at all, as is the case, for example, in the system of equations

$$\begin{aligned} \dot{x}_1 &= c + \sin(x_1 + x_2) + x_1 \\ \dot{x}_2 &= c + \sin(x_1 + x_2) - x_1 \end{aligned} \tag{3}$$

where $c \geq 2$ is a constant.

There are many important classes of systems that possess only one equilibrium. For example, consider the linear homogeneous system of equations given by

$$\dot{x} = A(t)x, \tag{LH}$$

where $A(t) = [a_{ij}(t)]$ denotes a real $n \times n$ matrix whose elements $a_{ij}(t)$ are continuous functions from R^+ into R (i.e., $a_{ij}: R^+ \to R$). The system (LH) has a unique equilibrium at the origin ($x_e = (x_1, x_2)^T = (0, 0)^T = 0$) if $A(t_0)$ is nonsingular for all $t_0 \geq 0$. Also, the autonomous system of equations

$$\dot{x} = f(x) \tag{A}$$

where $f: R^n \rightarrow R^n$ is assumed to be continuously differentiable with respect to all of its arguments, and where

$$J(x_e) = \frac{\partial f}{\partial x}(x)\,|_{x=x_e} \tag{4}$$

denotes the $n \times n$ Jacobian matrix defined by $\partial f/\partial x = [\partial f_i/\partial x_j]$ has an isolated equilibrium at x_e if $f(x_e) = 0$ and $J(x_e)$ is nonsingular.

Unless otherwise stated, we shall assume henceforth that a given equilibrium point is an isolated equilibrium. Also, we shall assume, unless otherwise stated, that in a given discussion, the equilibrium of interest is located at the origin of R^n. This assumption can be made without any loss of generality. To see this, assume that $x_e \neq 0$ is an equilibrium of system (E) [i.e., $f(x_e, t) = 0$ for all $t \geq 0$]. Let $w = x - x_e$. Then $w = 0$ is an equilibrium of the transformed system

$$\dot{w} = F(w, t) \tag{5}$$

where

$$F(w, t) = f(w + x_e, t) \tag{6}$$

Since system (6) establishes a one-to-one correspondence between the solutions of system (E) and system (5), we may assume henceforth that system (E) possesses the equilibrium of interest located at the origin. The equilibrium $x_e = 0$ will sometimes be referred to as the trivial solution of system (E).

LYAPUNOV AND LAGRANGE STABILITY CONCEPTS

We now state and interpret several definitions of stability of an equilibrium point, in the sense of Lyapunov.

The equilibrium $x_e = 0$ of system (E) is stable if for every $\epsilon > 0$ and $t_0 \geq 0$, there exists a $\delta(\epsilon, t_0) > 0$ such that

$$|p(t, x_0, t_0)| < \epsilon \text{ for all } t \geq t_0 \tag{7}$$

whenever

$$|x_0| < \delta(\epsilon, t_0) \tag{8}$$

[In system (7) and system (8), $|\cdot|$ denotes any one of the equivalent norms on R^n.]

In Fig. 1 we depict the behavior of the solutions (motions) in the vicinity of a stable equilibrium for the case $x \in R^2$. The interpretation of this figure is that when $x_e = 0$ is stable, then by choosing the initial points in a sufficiently small spherical neighborhood, we can force the graph of the solution for $t \geq t_0$ to lie entirely inside a given cylinder.

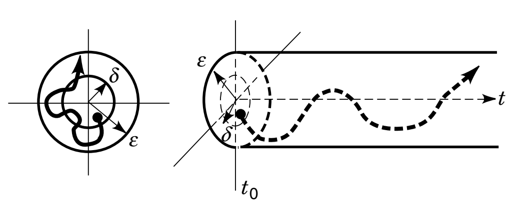

Figure 1. Qualitative behavior of a trajectory in the vicinity of a stable equilibrium.

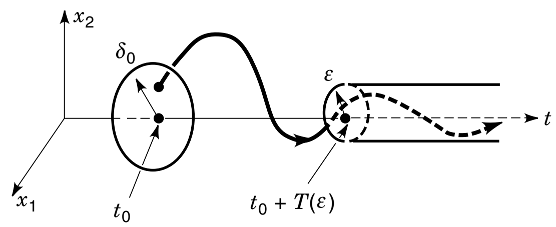

Figure 2. Qualitative behavior of a trajectory in the vicinity of an attractive equilibrium.

In the above definition of stability, δ depends on ϵ and t_0 [i.e., $\delta = \delta(\epsilon, t_0)$]. If δ is independent of t_0 [i.e., $\delta = \delta(\epsilon)$], then the equilibrium $x = 0$ of system (E) is said to be uniformly stable.

The equilibrium $x_e = 0$ of system (E) is said to be asymptotically stable if (1) it is stable, and (2) for every $t_0 \geq 0$ there exists an $\eta(t_0) > 0$ such that $\lim_{t\to\infty} p(t, x_0, t_0) = 0$ whenever $|x_0| < \eta$. Furthermore, the set of all $x_0 \in R^n$ such that $p(t, x_0, t_0) \to 0$ as $t \to \infty$ for some $t_0 \geq 0$ is called the domain of attraction of the equilibrium $x_e = 0$ of system (E). Also, if for system (E) condition (2) is true, then the equilibrium $x_e = 0$ is said to be attractive.

The equilibrium $x = 0$ of system (E) is said to be uniformly asymptotically stable if (1) it is uniformly stable, and (2) there is a $\delta_0 > 0$ such that for every $\epsilon > 0$ and any $t_0 \in R^+$, there exists a $T(\epsilon) > 0$, independent of t_0, such that $|p(t, x_0, t_0)| < \epsilon$ for all $t \geq t_0 + T(\epsilon)$ whenever $|x_0| < \delta_0$.

In Fig. 2 we depict pictorially property [system (2)] for uniform asymptotic stability. The interpretation of this figure is that by choosing the initial points in a sufficiently small spherical neighborhood at $t = t_0$, we can force the graph of the solution to lie inside a given cylinder for all $t > t_0 + T(\epsilon)$. Condition (2) can be rephrased by saying that there exists a $\delta_0 > 0$ such that $\lim_{t\to\infty} p(t + t_0, x_0, t_0) = 0$, uniformly in (x_0, t_0) for $t_0 \geq 0$ and for $|x_0| \leq \delta_0$.

In applications we are frequently interested in a special case of uniform asymptotic stability: the equilibrium $x_e = 0$ of system (E) is exponentially stable if there exists an $\alpha > 0$, and for $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$, such that $|p(t, x_0, t_0)| \leq \epsilon e^{\alpha(t-t_0)}$ for all $t \geq t_0$ whenever $|x_0| < \delta(\epsilon)$ and $t_0 \geq 0$.

In Fig. 3, the behavior of a solution in the vicinity of an exponentially stable equilibrium $x_e = 0$ is shown.

The equilibrium $x_e = 0$ of system (E) is said to be unstable if it is not stable. It is important to note that if $x_e = 0$ is an

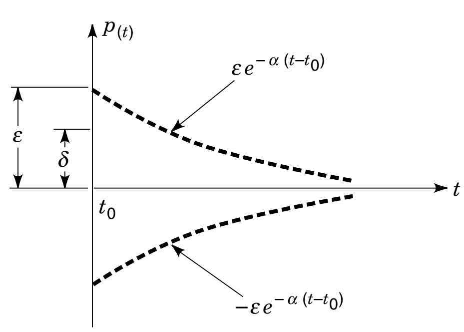

Figure 3. A trajectory envelope in the vicinity of an exponentially stable equilibrium.

unstable equilibrium, it still can happen that all the solutions tend to zero with increasing t. Thus, instability and attractivity of an equilibrium are compatible concepts. Note that the equilibrium $x_e = 0$ is necessarily unstable if every neighborhood of the origin contains initial points corresponding to unbounded solutions (i.e., solutions whose norm $|p(t, x_0, t_0)|$ grows to infinity on a sequence $t_m \to \infty$). However, it can happen that a system with unstable equilibrium $x_e = 0$ [see system (E)] may have only bounded solutions.

The above concepts pertain to local properties of an equilibrium. We now consider some global characterizations.

A solution $p(t, x_0, t_0)$ of system (E) is bounded if there exists a $\beta > 0$ such that $|p(t, x_0, t_0)| < \beta$ for all $t \geq t_0$, where β may depend on each solution. System (E) is said to possess Lagrange stability if for each $t_0 \geq 0$ and x_0 the solution $p(t, x_0, t_0)$ is bounded.

The solutions of system (E) are uniformly bounded if for any $\alpha > 0$ and $t_0 \in R^+$, there exists a $\beta = \beta(\alpha) > 0$ (independent of t_0) such that if $|x_0| < \alpha$, then $|p(t, x_0, t_0)| < \beta$ for all $t \geq t_0$.

The solutions of system (E) are uniformly ultimately bounded (with bound B) if there exists $B > 0$ and if corresponding to any $\alpha > 0$ and $t_0 \in R^+$, there exists a $T = T(\alpha)$ (independent of t_0) such that $|x_0| < \alpha$ implies that $|p(t, x_0, t_0)| < B$ for all $t \geq t_0 + T$.

In contrast to the boundedness properties given in the preceding three paragraphs, the concepts introduced earlier as well as those stated in the following are usually referred to as stability, respectively, instability, in the sense of Lyapunov.

The equilibrium $x_e = 0$ of system (E) is asymptotically stable in the large (or globally asymptotically stable) if it is stable and if every solution of system (E) tends to zero as $t \to \infty$. In this case, the domain of attraction of the equilibrium $x_e = 0$ of system (E) is all of R^n. Note that in this case, $x_e = 0$ is the only equilibrium of system (E).

The equilibrium $x_e = 0$ of system (E) is uniformly asymptotically stable in the large if (1) it is uniformly stable, and (2) for any $\alpha > 0$ and any $\epsilon > 0$ and $t_0 \in R^+$, there exists $T(\epsilon, \alpha) > 0$, independent of t_0, such that if $|x_0| < \alpha$, then $|p(t, x_0, t_0)| < \epsilon$ for all $t \geq t_0 + T(\epsilon, \alpha)$.

Finally, the equilibrium $x = 0$ of system (E) is exponentially stable in the large if there exists $\alpha > 0$ and for any $\beta > 0$, there exists $k(\beta) > 0$ such that $|p(t, x_0, t_0)| \leq k(\beta)|x_0|e^{\alpha(t-t_0)}$ for all $t \geq t_0$ whenever $|x_0| < \beta$. In the following, we cite several specific examples.

1. The scalar equation

$$\dot{x} = 0 \tag{9}$$

has for any initial condition $x(0) = x_0 = c$ the solution $p(t, c, 0) = c$. All solutions are equilibria for system (9). The trivial solution $x_e = 0$ is stable; in fact, it is uniformly stable.

2. The scalar equation

$$\dot{x} = ax,\ a > 0 \tag{10}$$

has for every initial condition $x(0) = x_0 = c$ the solution $p(t, c, 0) = ce^{at}$, and $x_e = 0$ is the only equilibrium of system (10). This equilibrium is unstable.

3. The scalar equation

$$\dot{x} = -ax,\ a > 0 \tag{11}$$

has for every initial condition $x(0) = x_0 = c$ the solution $p(t, c, 0) = ce^{-at}$, and $x_e = 0$ is the only equilibrium of system (11). This equilibrium is exponentially stable in the large.

4. The scalar equation

$$\dot{x} = \frac{-1}{t+1}x \tag{12}$$

has for every initial condition $x(t_0) = x_0 = c$, $t_0 \geq 0$, a unique solution of the form $p(t, c, t_0) = [(1 + t_0)c]/(t + 1)$, and $x_e = 0$ is the only equilibrium of system (12). This equilibrium is uniformly stable and asymptotically stable in the large, but it is not uniformly asymptotically stable.

5. By making use of the general properties of the solutions of linear autonomous homogeneous systems of equations given by

$$\dot{x} = Ax,\ t \geq 0 \tag{L}$$

where $A = [a_{ij}]$ is a real $n \times n$ matrix, the following has been established:

(i) The equilibrium $x_e = 0$ of system (L) is stable if all eigenvalues of A have nonpositive real parts and every eigenvalue of A that has a zero real part is a simple zero of the characteristic polynomial of A.

(ii) The equilibrium $x = 0$ of system (L) is asymptotically stable if and only if all eigenvalues of A have negative real parts. In this case, there exist constants $k > 0$, $\sigma > 0$ such that $|p(t, x_0, t_0)| \leq k|x_0|e^{-\sigma(t-t_0)}$ for all $t \geq t_0 \geq 0$.

LYAPUNOV FUNCTIONS

The general Lyapunov and Lagrange stability results for dynamical systems described by system (E) involve the existence of real-valued functions $v: D \to R$. In the case of local results (e.g., stability, instability, asymptotic stability, and exponential stability of an equilibrium $x_e = 0$), we shall usually only require that $D = B(h) \subset R^n$ for some $h > 0$, or $D = B(h) \times R^+$. (Recall that $B(h) = \{x \in R^n: |x| < h\}$ where $|x|$ denotes any one of the equivalent norms of x on R^n and $R^+ = [0, \infty)$.) On the other hand, in the case of global results [e.g., asymptotic stability in the large and exponential stability in the large of the equilibrium $x_e = 0$, and uniform boundedness of solutions of system (E)], we have to assume that $D = R^n$ or $D = R^n \times R^+$. Unless stated otherwise, we shall always assume that $v(0, t) = 0$ for all $t \in R^+$ [respectively, $v(0) = 0$].

Now let $p(t)$ be an arbitrary solution of system (E) and consider the function $t \mapsto v(p(t), t)$. If v is continuously differentiable with respect to all of its arguments, then we obtain, by the chain rule, the derivative of v with respect to t along the solutions of system (E), $\dot{v}_{(E)}$, as

$$\dot{v}_{(E)}(p(t), t) = \frac{\partial v}{\partial t}(p(t), t) + \nabla v(p(t), t)^T f(p(t), t) \tag{13}$$

where ∇v denotes the gradient vector of v with respect to x. Note that for a solution $p(t, x_0, t_0)$ of system (E) we have

$$v(p(t), t) = v(x_0, t_0) + \int_{t_0}^{t} \dot{v}_{(E)}(p(\tau, x_0, t_0), \tau)\, d\tau \tag{14}$$

The above observations motivate the following: $\dot{v}_{(E)}: R^n \times R^+ \to R$ (respectively, $\dot{v}_{(E)}: B(h) \times R^+ \to R$), defined by

$$\begin{aligned}\dot{v}_{(E)}(x,t) &= \frac{\partial v}{\partial t}(x,t) + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i}(x,t) f_i(x,t) \\ &= \frac{\partial v}{\partial t}(x,t) + \nabla v(x,t)^T f(x,t)\end{aligned} \tag{15}$$

is called the derivative of v, with respect to t, along the solutions of system (E).

It is important to note that in system (15), the derivative of v with respect to t, along the solutions of system (E), is evaluated without having to solve system (E). The significance of this will become clear later. We also note that when $v: R^n \to R$ (resp., $v: B(h) \to R$), then system (15) reduces to $\dot{v}_{(E)}(x, t) = \nabla v(x)^T f(x, t)$. Also, in the case of autonomous systems (A), if $v: R^n \to R$ (resp., $v: B(h) \to R$), we have

$$\dot{v}_{(A)}(x) = \nabla v(x)^T f(x) \tag{16}$$

Occasionally, we shall require only that v be continuous on its domain of definition and that it satisfy locally a Lipschitz condition with respect to x. In such cases we define the upper right-hand derivative of v with respect to t along the solutions of system (E) by

$$\dot{v}_{(E)}(x,t) = \limsup_{\theta \to 0+} (1/\theta)\{v[x + \theta \cdot f(x,t), t + \theta] - v(x,t)\} \tag{17}$$

When v is continuously differentiable, then system (17) reduces to system (15).

In characterizing v-functions of the type discussed above, we will employ Kamke comparison functions, which are defined as follows: a continuous function $\psi: [0, r_1] \to R^+$ (resp., $\psi: R^+ \to R^+$) is said to belong to class K (i.e., $\psi \in K$), if $\psi(0) = 0$ and if ψ is strictly increasing on $[0, r_1]$ (resp., on $[0, \infty)$). If $\psi: R^+ \to R^+$, if $\psi \in K$, and if $\lim_{r\to\infty} \psi(r) = \infty$, then ψ is said to belong to class KR.

We are now in a position to characterize v-functions in a variety of ways. In the following, we assume that $v: R^n \times R^+ \to R$ (resp., $v: B(h) \times R^+ \to R$), that $v(0, t) = 0$ for all $t \in R^+$, and that v is continuous.

a. v is said to be positive definite if for some $r > 0$, there exists a $\psi \in K$ such that $v(x, t) \geq \psi(|x|)$ for all $t \geq 0$ and for all $x \in B(r)$.
b. v is decrescent if there exists a $\psi \in K$ such that $|v(x, t)| \leq \psi(|x|)$ for all $t \geq 0$ and for all $x \in B(r)$ for some $r > 0$.
c. $v: R^n \times R^+ \to R$ is radially unbounded if there exists a $\psi \in KR$ such that $v(x, t) \geq \psi(|x|)$ for all $x \in R^n$ and for all $t \geq 0$.
d. v is negative definite if $-v$ is positive definite.
e. v is positive semidefinite if $v(x, t) \geq 0$ for all $x \in B(r)$ for some $r > 0$ and for all $t \geq 0$.
f. v is negative semidefinite if $-v$ is positive semidefinite.

The definitions corresponding to the above concepts when $v: R^n \to R$ or $v: B(h) \to R$ [where $B(h) \subset R^n$ for some $h > 0$] involve obvious modifications. We now consider some specific examples.

a. $v: R^3 \to R$ given by $v(x) = x^T x = x_1^2 + x_2^2 + x_3^2$ is positive definite and radially unbounded.
b. $v: R^3 \to R$ given by $v(x) = x_1^2 + (x_2 + x_3)^2$ is positive semidefinite, but not positive definite.
c. $v: R^2 \to R$ given by $v(x) = x_1^2 + x_2^2 - (x_1^2 + x_2^2)^3$ is positive definite but not radially unbounded.
d. $v: R^3 \to R$ given by $v(x) = x_1^2 + x_2^2$ is positive semidefinite but not positive definite.
e. $v: R^2 \to R$ given by $v(x) = x_1^4/(1 + x_1^4) + x_2^4$ is positive definite but not radially unbounded.
f. $v: R^2 \times R^+ \to R$ given by $v(x, t) = (1 + \cos^2 t)x_1^2 + 2x_2^2$ is positive definite, decrescent, and radially unbounded.
g. $v: R^2 \times R^+ \to R$ given by $v(x, t) = (x_1^2 + x_2^2)\cos^2 t$ is positive semidefinite and decrescent.
h. $v: R^2 \times R^+ \to R$ given by $v(x, t) = (1 + t)(x_1^2 + x_2^2)$ is positive definite and radially unbounded but not decrescent.
i. $v: R^2 \times R^+$ given by $v(x, t) = x_1^2/(1 + t) + x_2^2$ is decrescent and positive semidefinite but not positive definite.
j. $v: R^2 \times R^+ \to R$ given by $v(x, t) = (x_2 - x_1)^2(1 + t)$ is positive semidefinite but not positive definite or decrescent.

Of special interest are quadratic forms $v: R^n \to R$ given by

$$v(x) = x^T Bx = \sum_{i,k=1}^{n} b_{ik} x_i x_k \tag{18}$$

where $B = [b_{ij}]$ is a real, symmetric $n \times n$ matrix. Since B is symmetric, it is diagonizable and all its eigenvalues are real. Let λ_m and λ_M denote the smallest and largest eigenvalues of B and let $|x|$ denote the Euclidean norm of x. It has been shown that

$$\lambda_m |x|^2 \leq v(x) \leq \lambda_M |x|^2 \tag{19}$$

for all $x \in R^n$. From system (19) these facts follow now immediately:

a. v is definite (i.e., either positive definite or negative definite) if and only if all eigenvalues are nonzero and have the same sign.
b. v is semidefinite (i.e., either positive semidefinite or negative semidefinite) if and only if the nonzero eigenvalues of B have the same sign.
c. v is indefinite (i.e., in every neighborhood of the origin $x = 0$, v assumes positive and negative values) if and only if B possesses both positive and negative eigenvalues.

It has also been shown that v given by system (18) is positive definite (and radially unbounded) if and only if all principal minors of the matrix B are positive, that is, if and only if

$$\det \begin{bmatrix} b_{11} & \cdots & b_{1k} \\ \vdots & & \vdots \\ b_{k1} & \cdots & b_{kk} \end{bmatrix} > 0, \quad k = 1, \ldots, n$$

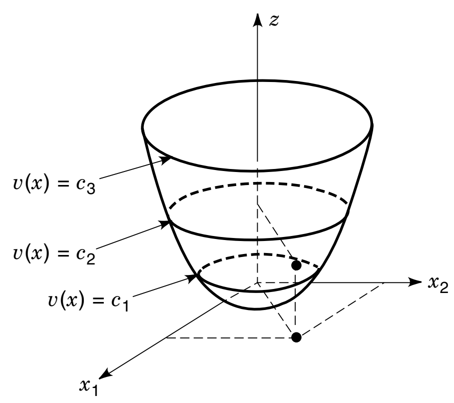

Figure 4. Surface described by a quadratic form.

Furthermore, v given by system (18) is negative definite if and only if

$$(-1)^k \det \begin{bmatrix} b_{11} & \cdots & b_{1k} \\ \vdots & & \vdots \\ b_{k1} & \cdots & b_{kk} \end{bmatrix} > 0, \quad k = 1, \ldots, n$$

It turns out that quadratic forms [system (18)] have some interesting geometric properties, as is shown next. Let $n = 2$ and assume that both eigenvalues of B are positive, which means that v is positive definite and radially unbounded. In R^3, the surface determined by the equation

$$z = v(x) = x^T Bx \tag{20}$$

describes a cup-shaped surface as shown in Fig. 4. Note in this figure that corresponding to every point on this cup-shaped surface there exists one and only one point in the x_1x_2 plane. Note also that the loci defined by $C_i = \{x \in R^2 : v(x) = c_i \geq 0\}$, c_i = constant, determine closed curves in the x_1x_2 plane as shown in Fig. 5. These are called level curves. Note that $C_0 = \{0\}$ corresponds to the case $z = c_0 = 0$. Further, note also that this function v can be used to cover the entire R^2 plane with closed curves by selecting for z all values in R^+.

In the more general case, when $x \in R^n$, $n > 2$, and B is positive definite, the preceding discussion concerning quadratic forms [system (18)] still holds; however, in this case, the closed curves C_i must be replaced by closed hypersurfaces in R^n and simple visualizations as shown in Figs. 4 and 5 are no longer possible.

LYAPUNOV STABILITY RESULTS—MOTIVATION

Before presenting the Lyapunov and Lagrange stability results, we will give geometric interpretations for some of these. To this end we consider dynamical systems determined by two first-order ordinary differential equations of the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned} \tag{21}$$

and we assume that for every (x_0, t_0), $t_0 \geq 0$, system (21) has a unique solution $p(t, x_0, t_0)$ with $p(t_0, x_0, t_0) = x_0$. We also assume that $(x_1, x_2)^T = (0, 0)^T$ is the only equilibrium in $B(h)$ for some h > 0.

Next, let v be a positive definite, continuously differentiable function with nonvanishing gradient ∇v on $0 < |x| \leq h$. Then $v(x) = c$, $c \geq 0$, defines for sufficiently small constants $c > 0$ a family of closed curves C_i, which cover the neighborhood $B(h)$ as shown in Fig. 6. Note that the origin $x = 0$ is located in the interior of each curve and $C_0 = \{0\}$.

Now suppose that all solutions (motions) of system (21) originating from points on the circular disk $|x| \leq r_1 < h$ cross the curves $v(x) = c$ from the exterior toward the interior when we proceed along these solutions in the direction of increasing values of t. Then we can conclude that these solutions approach the origin as t increases (i.e., the equilibrium $x = 0$ in this case is asymptotically stable).

In terms of the given v function, we have the following interpretation. For a given solution $p(t, x_0, t_0)$ to cross the curve $v(x) = r$, $r = v(x_0)$, the angle between the outward normal vector $\nabla v(x_0)$ and the derivative of $p(t, x_0, t_0)$ at $t = t_0$ must be greater than $\pi/2$, that is,

$$\dot{v}_{(21)}(x_0) = \nabla v(x_0)^T f(x_0) < 0$$

For this to happen at all points, we must have $\dot{v}_{(21)}(x) < 0$ for $0 < |x| \leq r_1$. The same results can be arrived at from an analytic point of view. The function $V(t) \stackrel{\Delta}{=} v[p(t, x_0, t_0)]$ decreases monotonically as t increases. This implies that the derivative $\dot{v}[p(t, x_0, t_0)]$ along the solution $p(t, x_0, t_0)$ must be negative definite in $B(r)$ for $r > 0$ sufficiently small.

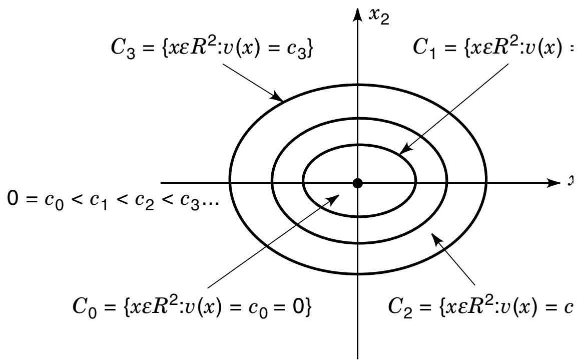

Figure 5. Level curves determined by a quadratic form.

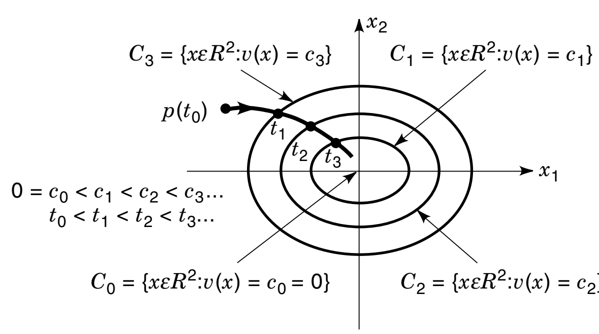

Figure 6. Solution (motion) near an asymptotically stable equilibrium.

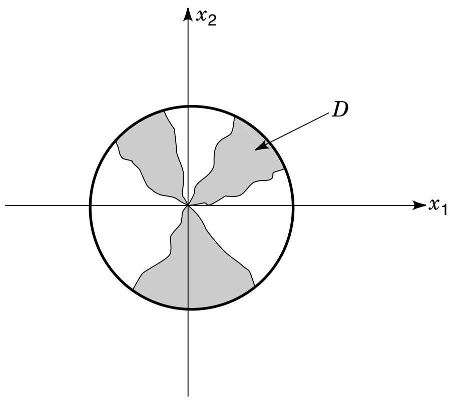

Figure 7. Instability of an equilibrium.

Proceeding, let us next assume that system (21) has only one equilibrium, $x_e = 0$, and that v is positive definite and radially unbounded. In this case, the relation $v(x) = c$, $c \in R^+$, can be used to cover all of R^2 by closed curves of the type shown in Fig. 6. If for arbitrary initial conditions (x_0, t_0), the solution of system (21), $p(t, x_0, t_0)$, behaves as already discussed, then it follows that the derivative of v along this solution, $\dot{v}(p[t, x_0, t_0,)]$, will be negative definite in R^2.

The foregoing discussion was given in terms of an arbitrary solution of system (21). This suggests the following results:

1. If there exists a positive definite function v such that $\dot{v}_{(21)}$ is negative definite, then the equilibrium $x_e = 0$ of system (21) is asymptotically stable.
2. If there exists a positive definite and radially unbounded function v such that $\dot{v}_{(21)}$ is negative definite for all $x \in R^2$, then the equilibrium $x_e = 0$ of system (21) is asymptotically stable in the large.

Continuing by making reference to Fig. 7, let us now assume that we can find for system (21) a continuously differentiable function $v: R^2 \to R$ that is indefinite and that has the properties discussed in the following. Since v is indefinite, there exist in each neighborhood of the origin points for which $v > 0$, $v < 0$, and $v(0) = 0$. Confining our attention to $B(k)$, where $k > 0$ is sufficiently small, we let $D = \{x \in B(k): v(x) < 0\}$. Note that D may consist of several subdomains. The boundary of D, ∂D, as shown in Fig. 7, consists of points in $\partial B(k)$ and of points determined by $v(x) = 0$. Let us assume that in the interior of D, v is bounded. Suppose that $\dot{v}_{(21)}$ is negative definite in D and that $p(t)$ is a solution of system (21) that originates somewhere on the boundary of D [i.e., $p(t_0, x_0, t_0) = x_0 \in \partial D$] with $v(x_0) = 0$. Then this solution will penetrate the boundary of D at points where $v = 0$ as t increases, and it can never again reach a point where $v = 0$. Indeed, as t increases, this solution will penetrate the set of points determined by $|x| = k$, since by assumption, $\dot{v}_{(21)} < 0$ along this solution and since $v < 0$ in D. But this shows that the equilibrium $x_e = 0$ of system (21) is unstable. This discussion leads us yet to another conjecture:

3. Assume there exists a continuously differentiable function $v: R^2 \to R$ with the following properties:
   (i) there exist points x arbitrarily close to the origin such that $v(x) < 0$, which form the domain D which is bounded by the set of points determined by $v = 0$ and the disk $|x| = k$;
   (ii) in the interior of D, v is bounded; and
   (iii) in the interior of D, $\dot{v}_{(21)}$ is negative.

   Then the equilibrium $x_e = 0$ of system (21) is unstable.

THE PRINCIPAL LYAPUNOV AND LAGRANGE STABILITY THEOREMS

It turns out that results of the type presented in the previous section for system (21) are true for general systems given by system (E). This is true for the case of Lyapunov stability and Lagrange stability. These results comprise the Lyapunov Method, or the Second Method of Lyapunov, or the Direct Method of Lyapunov of qualitative analysis of dynamical systems. The reason for the latter name is clear: results of the kind considered here allow us to make qualitative statements about entire families of solutions of system (E) without actually solving this equation.

In the following, we summarize most of the important Lyapunov and Lagrange stability results for dynamical systems determined by system (E). Their proofs can be found in many texts on ordinary differential equations or on the stability of dynamical systems. We shall cite some of these sources when discussing the literature on the present subject.

In each of the following statements, we shall assume the existence of a continuously differentiable function $v: B(h) \times R^+ \to R$ for some $h > 0$, or $v: R^n \times R^+ \to R$, as needed.

1. If v is positive definite and $\dot{v}_{(E)}$ is negative semidefinite (or identically zero), then the equilibrium $x_e = 0$ of system (E) is stable.

2. If v is positive definite and decrescent and $\dot{v}_{(E)}$ is negative semidefinite (or identically zero), then the equilibrium $x_e = 0$ of system (E) is uniformly stable.

3. If v is positive definite and decrescent and $\dot{v}_{(E)}$ is negative definite, then the equilibrium $x_e = 0$ of system (E) is uniformly asymptotically stable.

4. If v is positive definite, decrescent, and radially unbounded and $\dot{v}_{(E)}$ is negative definite for all $(x, t) \in R^n \times R^+$, then the equilibrium $x_e = 0$ of system (E) is uniformly asymptotically stable in the large.

5. If there exist three positive constants c_1, c_2, c_3 such that

$$\begin{aligned} c_1|x|^2 \le v(x,t) \le c_2|x|^2 \\ \dot{v}_{(E)}(x,t) \le -c_3|x|^2 \end{aligned} \tag{22}$$

for all $t \in R^+$ and all $x \in B(r)$ for some $r > 0$, then the equilibrium $x_e = 0$ of system (E) is exponentially stable.

6. If there exist three positive constants c_1, c_2, c_3 such that system (22) holds for all $t \in R^+$ and all $x \in R^n$, then the equilibrium $x_e = 0$ of system (E) is exponentially stable in the large.

7. If v is decrescent and $\dot{v}_{(E)}$ is positive definite (resp., negative definite) and if in every neighborhood of the origin there are points x such that $v(x, t_0) > 0$ (resp., $v(x, t_0) < 0$), then the equilibrium $x_e = 0$ of system (E) is unstable (at $t = t_0 \ge 0$).

8. Assume that v is bounded on $D = \{(x, t): x \in B(h), t \ge t_0\}$ and satisfies the following: (i) $\dot{v}_{(E)}(x, t) = \lambda v(x, t) + w(x, t)$,

where $\lambda > 0$ is a constant and $w(x, t)$ is either identically zero or positive semidefinite; (ii) in the set $D_1 = \{(x, t): t = t_1, x \in B(h_1)\}$ for fixed $t_1 \geq t_0$ and with arbitrarily small h_1, there exist values x such that $v(x, t_1) > 0$. Then the equilibrium $x_e = 0$ of system (E) is unstable.

9. Assume that v satisfies the following properties:

(i) For every $\epsilon > 0$ and for every $t \geq 0$, there exist points $\bar{x} \in B(\epsilon)$ such that $v(\bar{x}, t) < 0$. We call the set of all points (x, t) such that $x \in B(h)$ and such that $v(x, t) < 0$ the "domain $v < 0$." This domain is bounded by the hypersurfaces that are determined by $|x| = h$ and $v(x, t) = 0$, and it may consist of several component domains.

(ii) In at least one of the component domains D of the "domain $v < 0$," v is bounded from below and $0 \in \partial D$ for all $t \geq 0$.

(iii) In the domain D, $\dot{v}_{(E)} \leq -\psi(|v|)$, where $\psi \in K$.

Then the equilibrium $x_e = 0$ of system (E) is unstable.

The next two results are typical of Lagrange-type stability results. In both of these results we assume that v is continuously differentiable and is defined on $|x| \geq R$, where R may be large, and $0 \leq t < \infty$.

10. Assume there exist $\psi_1, \psi_2 \in KR$ such that $\psi_1(|x|) \leq v(x, t) \leq \psi_2(|x|)$ and $\dot{v}_{(E)}(x, t) \leq 0$ for all $|x| \geq R$ and for all $0 \leq t < \infty$. Then the solutions of system (E) are uniformly bounded.

11. Assume there exist $\psi_1, \psi_2 \in KR$ and $\psi_3 \in K$ such that $\psi_1(|x|) \leq v(x, t) \leq \psi_2(|x|)$ and $\dot{v}_{(E)}(x, t) \leq -\psi_3(|x|)$ for all $|x| \geq R$ and $0 \leq t < \infty$. Then the solutions of system (E) are uniformly ultimately bounded.

We now apply some of the above results to some specific examples.

The system given by

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -x_2 - e^{-t}x_1 \tag{23}$$

has an equilibrium at $(x_1, x_2)^T = (0, 0)^T$. We choose for system (23) the positive definite function $v(x_1, x_2, t) = x_1^2 + e^t x_2^2$ and obtain $\dot{v}_{(23)}(x_1, x_2, t) = -e^t x_2^2$ which is negative semidefinite. The result in item 1 above applies and we conclude that the equilibrium $x_e = 0$ of system (23) is stable.

We consider the simple pendulum considered earlier which is described by the equations

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -k \sin x_1 \tag{24}$$

where $k > 0$ is a constant. As noted earlier, system (24) has an isolated equilibrium at $(x_1, x_2)^T = (0, 0)^T$. Choose $v(x_1, x_2) = \frac{1}{2}x_2^2 + k\int_0^{x_1} \sin\, \eta d\eta$, which is continuously differentiable and positive definite. We note that since v is independent of t, it is automatically decrescent. Furthermore, $\dot{v}_{(24)}(x_1, x_2) = (k \sin x_1)\dot{x}_1 + x_2\dot{x}_2 = (k \sin x_1)x_2 + x_2(-k \sin x_1) = 0$. The result in item 2 above applies and we conclude that the equilibrium $x_e = 0$ of system (24) is uniformly stable.

The system given by

$$\begin{aligned} \dot{x}_1 &= (x_1 - k_2 x_2)(x_1^2 + x_2^2 - 1) \\ \dot{x}_2 &= (k_1 x_1 + x_2)(x_1^2 + x_2^2 - 1) \end{aligned} \tag{25}$$

has an isolated equilibrium at $(x_1, x_2)^T = (0, 0)^T$. For system (25) we choose $v(x) = k_1x_1^2 + k_2x_2^2$ and obtain $\dot{v}_{(25)}(x_1, x_2) = 2(k_1x_1^2 + k_2x_2^2)(x_1^2 + x_2^2 - 1)$. If $k_1 > 0$, $k_2 > 0$, then v is positive definite (and decrescent) and $\dot{v}_{(25)}$ is negative definite over the domain $x_1^2 + x_2^2 < 1$. Accordingly, the result in item 3 above applies and we conclude that the equilibrium $(x_1, x_2)^T = (0, 0)^T$ is uniformly asymptotically stable.

The system given by

$$\begin{aligned} \dot{x}_1 &= x_2 + cx_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= -x_1 + cx_2(x_1^2 + x_2^2) \end{aligned} \tag{26}$$

where c is a real constant, has only one equilibrium, which is located at the origin. For system (26) we choose the positive definite, decrescent, and radially unbounded function $v(x_1, x_2) = x_1^2 + x_2^2$ to obtain $\dot{v}_{(26)}(x_1, x_2) = 2c(x_1^2 + x_2^2)^2$. When $c = 0$, the result in item 2 above applies and we conclude that the equilibrium $(x_1, x_2)^T = (0, 0)^T$ is uniformly stable. If $c < 0$, then the result in item 4 above applies and we conclude that the trivial solution of system (26) is uniformly asymptotically stable in the large. If $c > 0$, then the result in item 7 above applies and we conclude that the trivial solution of system (26) is unstable.

For the system

$$\begin{aligned} \dot{x}_1 &= -a(t)x_1 - bx_2 \\ \dot{x}_2 &= bx_1 - c(t)x_2 \end{aligned} \tag{27}$$

b is a real constant and a and c are real and continuous functions defined for $t \geq 0$ and satisfying $a(t) \geq \delta > 0$ and $c(t) \geq \delta > 0$ for all $t \geq 0$. We assume that a, b and c are such that $x = 0$ is the only equilibrium for system (27). If we choose $v(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$, then $\dot{v}_{(27)}(x, t) = -a(t)x_1^2 - c(t)x_2^2 \leq -\delta(x_1^2 + x_2^2)$ for all $(x_1, x_2)^T \in R^2$ and for all $t \geq 0$. The result in item 6 above applies and we conclude that the equilibrium $(x_1, x_2)^T = (0, 0)^T$ of system (27) is exponentially stable in the large.

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 + x_1x_2^4 \\ \dot{x} &= x_1 + x_2 - x_1^2x_2 \end{aligned} \tag{28}$$

which has an isolated equilibrium $(x_1, x_2)^T = (0, 0)^T$. Choosing $v(x_1, x_2) = (x_1^2 - x_2^2)/2$, we obtain $\dot{v}_{(28)}(x_1, x_2) = \lambda v(x_1, x_2) + w(x_1, x_2)$, where $w(x_1, x_2) = x_1^2x_2^4 + x_1^2x_2^2$ and $\lambda = 2$. The result in item 8 above applies and we conclude that the equilibrium $(x_1, x_2)^T = (0, 0)^T$ is unstable.

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= x_1 - x_1 + x_1x_2 \end{aligned} \tag{29}$$

which has an isolated equilibrium at the origin $(x_1, x_2)^T = (0, 0)^T$. Choosing $v(x_1, x_2) = -x_1x_2$ we obtain $\dot{v}_{(29)}(x_1, x_2) = -x_1^2 - x_2^2 - x_1^2x_2$. Let $D = \{(x_1, x_2)^T \in R^2 : x_1 > 0, x_2 > 0, \text{ and } x_1^2 + x_2^2 < 1\}$. Then for all $(x_1, x_2)^T \in D$, $v(x_1, x_2) < 0$ and $\dot{v}_{(29)}(x_1, x_2) < 2v(x_1, x_2)$. We see that the result in item 9 above applies and conclude that the equilibrium $(x_1, x_2)^T = (0, 0)^T$ is unstable.

Consider the system

$$\dot{x} = -x - \sigma,\ \dot{\sigma} = -\sigma - f(\sigma) + x \tag{30}$$

where $f(\sigma) = \sigma(\sigma^2 - 6)$. This system has three isolated equilibria located at $x = \sigma = 0$, $x = -\sigma = 2$, and $x = -\sigma = -2$. Choosing the radially unbounded and decrescent function $v(x, \sigma) = \frac{1}{2}(x^2 + \sigma^2)$, we obtain $\dot{v}_{(30)}(x, \sigma) = -x^2 - \sigma^2(\sigma^2 - 5) \leq -x^2 - (\sigma^2 - \frac{5}{2})^2 + \frac{25}{4}$. Note also that $\dot{v}_{(30)}(x, \sigma)$ is negative for all (x, σ) such that $x^2 + \sigma^2 > R^2$, where, for example, $R = 10$ is an acceptable choice. Therefore, in accordance with the results given in items 10 and 11 above, all solutions of system (30) are uniformly bounded, in fact, uniformly ultimately bounded.

We conclude the present section by noting that the results given above in items 1–11 are also true when v is continuous, rather than continuously differentiable. In this case, $\dot{v}_{(E)}$ is interpreted as in system (17).

SOME EXTENSIONS AND FURTHER RESULTS

The body of work concerned with the Lyapunov Method is vast. In the following, we present a few additional rather well-known results.

For the case of autonomous systems given by

$$\dot{x} = f(x) \tag{A}$$

$f(0) = 0$, it is sometimes possible to relax the conditions on $\dot{v}_{(A)}$ (given in the previous section) when investigating the asymptotic stability of the equilibrium $x_e = 0$, by insisting that $\dot{v}_{(A)}$ be only negative semidefinite. In doing so, we require the following concept: a set $\Gamma \subset R^n$ is said to be invariant with respect to system (A) if every solution of system (A) starting in Γ remains in Γ for all time.

The following theorem is one of the results that comprise the Invariance Theory in the stability analysis of dynamical systems determined by system (A): Assume that there exists a continuously differentiable, positive definite, and radially unbounded function $v: R^n \to R$ such that

(i) $\dot{v}_{(A)}(x) \leq 0$ for all $x \in R^n$, and
(ii) the set $\{0\}$ is the only invariant subset of the set $E = \{x \in R^n : \dot{v}_{(A)}(x) = 0\}$.

Then the equilibrium $x_e = 0$ of system (A) is asymptotically stable in the large.

We apply the above invariance theorem in the analysis of the Lienard Equation given by

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -f(x_1)x_2 - g(x_1) \tag{31}$$

where it is assumed that f and g are continuously differentiable for all $x_1 \in R$, $g(x_1) = 0$ if and only if $x_1 = 0$, $x_1 g(x_1) > 0$ for all $x_1 \neq 0$ and $x_1 \in R$,

$$\lim_{|x_1| \to \infty} \int_0^{x_1} g(\eta)\, d\eta = \infty$$

and $f(x_1) > 0$ for all $x_1 \in R$. Under these assumptions, the origin $(x_1, x_2)^T = (0, 0)^T$ is the only equilibrium of system (31).

Let us now choose the v function

$$v(x_1, x_2) = \frac{1}{2}x_2^2 + \int_0^{x_1} g(\eta)\, d\eta$$

which is positive definite and radially unbounded. Along the solutions of system (31) we have $\dot{v}_{(31)}(x_1, x_2) = -x_2^2 f_2(x_1) \leq 0$ for all $(x_1, x_2)^T \in R^2$. It is easy to see that in the present case the set E is the entire x_1-axis and that the largest invariant subset of the set E with respect to system (31) is the set $\{(0, 0)^T\}$. In view of the Invariance Stability Theorem given above, the origin $(x_1, x_2)^T = (0, 0)^T$ is asymptotically stable in the large.

The power, generality, and elegance of the Lyapunov Method must be obvious by now. However, this method has also weaknesses, the greatest drawback being that there exist no rules for choosing v-functions (Lyapunov functions). However, for the case of linear systems given by

$$\dot{x} = Ax \tag{L}$$

it is possible to construct Lyapunov functions in a systematic manner, in view of the following result.

Assume that the matrix A has no eigenvalues on the imaginary axis. Then there exists a Lyapunov function v of the form

$$v(x) = x^T Bx, B = B^T \tag{32}$$

whose derivative $\dot{v}_{(L)}$, given by

$$\dot{v}_{(L)} = -x^T Cx$$

where

$$-C = A^T B + BA \tag{33}$$

is definite (i.e., negative definite or positive definite).

In particular, the above results states that if all eigenvalues of A have negative real parts (i.e., the matrix A is stable), then for system (L), our earlier Lyapunov result for asymptotic stability in the large constitutes also the necessary conditions for asymptotic stability. In the same spirit, an instability results for system (L) can also be established. We will not pursue this, however.

In view of the above result, if for example, all eigenvalues of A have negative real parts, then the v-function [system (32)] is easily constructed by assuming a positive definite matrix $C = C^T$ and by solving the Lyapunov matrix equation [(system 33)] for the $n(n + 1)/2$ unknown elements of the symmetric matrix B (which in this case will be positive definite).

To simplify matters, we consider in the following autonomous systems described by

$$\dot{x} = f(x) \tag{A}$$

and we assume that $x_e = 0$ is an equilibrium of system (A) [i.e., $f(0) = 0$]. Now, when the origin is not the only equilibrium of system (A) and if $x_e = 0$ is asymptotically stable, then $x_e = 0$ cannot possibly be globally asymptotically stable. [There may be other reasons why an asymptotically stable equilibrium $x_e = 0$ of system (A) might not be globally asymptotically stable.] Under such conditions, it is of great interest to determine an estimate of the domain of attraction of the equilibrium $x_e = 0$ of system (A).

Now for purposes of discussion, let us assume that for system (A) there exists a Lyapunov function v that is positive definite and radially unbounded. Also, let us assume that over some domain $D \subset R^n$ containing the origin, $\dot{v}_{(A)}(x)$ is negative, except at the origin, where $\dot{v}_{(A)} = 0$. Let $C_i = \{x \in R^n : v(x) \leq c_i\}$, $c_i > 0$. Using similar reasoning as was done in the analysis of the system (21), we can now show that as long as $C_i \subset D$, C_i will be a subset of the domain of attraction of $x_e = 0$. Thus, if $c_i > 0$ is the largest number for which this is true, then it follows that C_i will be contained in the domain of attraction of $x_e = 0$. The set C_i obtained in this manner will be the best estimate for the domain of attraction of $x_e = 0$ that can be obtained using our particular choice of v-function.

Above we pointed out that for system (L) there actually exist converse Lyapunov (asymptotic stability and instability) theorems. It turns out that for virtually every Lyapunov and Lagrange Stability Theorem given earlier, a converse can be established. Unfortunately, these Lyapunov converse theorems are of not much help in constructing v-functions in specific cases. For purposes of illustration, we cite in the following an example of such a converse theorem.

If f and $\partial f/\partial x$ are continuous on the set $B(r) \times R^+$ for some $r > 0$, and if the equilibrium $x_e = 0$ of system (E) is uniformly asymptotically stable, then there exists a Lyapunov function v which is continuously differentiable on $B(r_1) \times R^+$ for some $r_1 > 0$ such that v is positive definite and decrescent, and such that $\dot{v}_{(E)}$ is negative definite.

We conclude this section by addressing the following question: under what conditions does it make sense to linearize a nonlinear system about an equilibrium $x_e = 0$ and then deduce the stability properties of $x_e = 0$ from the corresponding linear system? Results that answer questions of this kind comprise Lyapunov's First Method or Lyapunov's Indirect Method.

To simplify our discussion, we consider autonomous systems (A),

$$\dot{x} = f(x) \tag{A}$$

we assume that f is continuously differentiable, and we assume that $f(0) = 0$, which means that $x_e = 0$ is an equilibrium for system (A).

A linearization process of system (A) about the equilibrium $x_e = 0$ results in the representation of system (A) by

$$\dot{x} = Ax + F(x) \tag{34}$$

where

$$A = \frac{\partial f}{\partial x}(0)$$

denotes the Jacobian of $f(x)$ evaluated at $x = 0$, and where

$$\lim_{|x| \to 0} \frac{|F(x)|}{|x|} = 0 \tag{35}$$

Associated with system (34) [respectively, system (A)], we have the system

$$\dot{y} = Ay \tag{36}$$

which is called the linearization of system (A).

Now suppose that the matrix A in system (36) is stable (i.e., all eigenvalues of A have negative real parts). According to results given above, we can construct in this case a Lyapunov function of the form (32) for system (36). Utilizing this Lyapunov function in the analysis of the nonlinear system (34) [and hence, of the original system (A)], and applying Lyapunov's asymptotic stability theorem that was presented earlier, the following result is established:

Assume that for the real $n \times n$ matrix A all eigenvalues have negative real parts and let $F: R^n \to R^n$ be continuous and satisfy system (35). Then the equilibrium $x_e = 0$ of system (34) [and hence, of system (A)] is asymptotically stable.

An instability theorem in the spirit of the above result has also been established. In fact, for system (E), theorems along the lines of the above results have been established as well.

We close the present section by considering the following version of the Lienard equation,

$$\dot{x}_1 = x_2, \; \dot{x}_2 = -x_1 - f(x_1)x_2 \tag{37}$$

where f is assumed to be continuously differentiable and $f(0) > 0$. The origin is clearly an equilibrium of system (37),

$$J(0) = A = \begin{bmatrix} 0 & 1 \\ -1 & -f(0) \end{bmatrix}$$

and the eigenvalues of A are given by $\lambda_1, \lambda_2 = [-f(0) \pm \sqrt{f(0)^2 - 4}]/2$. These have clearly negative real parts. Furthermore, it is easily verified that system (35) is satisfied. It follows that the trivial solution of system (37) is asymptotically stable. It must be emphasized, however, that this analysis by the First Method of Lyapunov does not yield any information whatsoever about the domain of attraction of the equilibrium $x_e = 0$ of system (37). This is true, in general.

SOME NOTES AND REFERENCES

Reference 1 is a translation of a paper from the Russian that originally had appeared in 1893 in a mathematics journal in Kharkow (*Comm. Soc. Math. Kharkow*). In this paper, A. M. Lyapunov developed a highly original approach for the stability analysis of an equilibrium of systems described by ordinary differential equations, which today bears his name under several variants: Lyapunov's Second Method, Lyapunov's Direct Method, Lyapunov's Method, and so forth. (In the present article, Lyapunov's First Method is also included.) It is interesting to note that since the motivation for his work was an analysis of the motions of planets, Lyapunov was actually more interested in the stability (and instability) of an equilibrium, rather than in asymptotic stability. For an account of early work on this subject, the reader may want to consult the book by Bennett (2), and some of the sources cited therein.

Since 1893, the Lyapunov approach has been extended, generalized, and improved in numerous ways, and the literature on this subject has experienced phenomenal growth, especially in recent times. Results that are in the spirit of those presented herein have been discovered for general dynamical systems (3–5), and for more specific classes of infinite dimensional systems (6) and finite dimensional systems (7,8).

Perhaps the greatest driving force behind the development of Lyapunov's Method was the significant progress that has been made since World War II in feedback control systems.

For a brief description of this, the reader may wish to consult Ref. 9 and the sources cited in that paper.

One of the early important problems in feedback control concerns the absolute stability of regulator systems. It is fair to say that most of the progress that was made toward solving this class of problems was accomplished by means of Lyapunov's approach (10–12). Another important class of feedback problems treated primarily by the Lyapunov Method was the systematic stability analysis of complex, large-scale dynamical systems (13–15), with specific applications in such diverse areas as power systems (16) and artificial neural networks (17). These two classes of systems are only a small sample where the Lyapunov approach has been effective in stability analysis. There are many other such classes, too numerous to cite here. The reader may want to consult some of the contemporary texts in control systems to obtain additional insights into this subject (18–20).

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LYAPUNOV STABILITY. See LYAPUNOV METHODS.