J. Webster (ed.), *Wiley Encyclopedia of Electrical and Electronics Engineering*

MINMAX TECHNIQUES

In real life, people need to make decisions based on facts such as measurements, scheduling, short- and long-term forecasting, and guessing. These decisions may be related to management, security, economics, and education, and so forth. However, in the last few years, the decision process has become more complex due to the large amount of information associated with each step of the decision-making process.

Usually, in order to help human operators make the right decision at the right time, there is a collection of operational instructions, standards, computer programs, and other information. However, sometimes these regulations and standards represent the main drawback for good and reliable decision making, because they have been obtained from the analysis of a specific situation or a certain particular study and hence do not apply to a different event. In this case the available information tends to push the operator to make a wrong decision. To cope with the decision-making problem in a complex environment, new mathematical tools have been developed to help create more flexible, friendly, and easy-to-build computerized decision-making systems. Among these tools, the fuzzy set theory plays a very important role today when the values assumed by the variables are linguistic values such as "small," "big," "warm," "cold," and "close." In the last few years, the reported new applications of this theory have reached an impressive number of areas. Industrial automation, equipment automation, expert systems, medical diagnostics, and control systems are some of the areas to which fuzzy sets are intensively applied today. The fuzzy set theory (1) was proposed as a step toward modeling the pervasive imprecision of the real world.

This article presents the theory of fuzzy sets and develops the fuzzy technique. *Fuzzy technique* is the name given to the process through which fuzzy set theory is applied to problems of the real world. Initially, some basic aspects of ordinary sets are presented, and then brief concepts of fuzzy set theory are addressed. Next, the fuzzy technique is presented and sequentially developed. Finally, an illustrative example of the fuzzy technique is presented.

Basic Operations With Ordinary Sets

Let U be a set of elements representing the universe of discourse and A and B subsets of U. Table 1 presents well-known operations and properties of these two subsets. One of these properties is the *intersection* ($A \cap B$), which can be expressed in linguistic terms by the conjunction ***and***. Another property, the *union* between two sets ($A \cup B$), can be expressed by the conjunction ***or***.

It is possible to establish a relationship between the properties *intersection* and *union* proposed in Table 1 and the classical Boolean arithmetic. *Intersection* can be expressed by a Boolean product, while the *union* is a Boolean sum, as shown in Eqs. (1) to (4), and Tables 2 and 3.

$$x \in (\boldsymbol{A} \cap \boldsymbol{B}) \quad \text{if} \quad x \in \boldsymbol{A} \text{ and } x \in \boldsymbol{B} \tag{1}$$

Table 1: Basic Operations and Properties

Properties of $\oslash$ and **U**
$\boldsymbol{A} \cup \oslash = \boldsymbol{A}$ $\boldsymbol{A} \cap \oslash = \oslash$ $\boldsymbol{A} \cup \boldsymbol{U} = \boldsymbol{U}$ $\boldsymbol{A} \cap \boldsymbol{U} = \boldsymbol{A}$
Idempotent Properties
$\boldsymbol{A} \cup \boldsymbol{A} = \boldsymbol{A}$ $\boldsymbol{A} \cap \boldsymbol{A} = \boldsymbol{A}$
Commutative Properties
$\boldsymbol{A} \cup \boldsymbol{B} = \boldsymbol{B} \cup \boldsymbol{A}$ $\boldsymbol{A} \cap \boldsymbol{B} = \boldsymbol{B} \cap \boldsymbol{A}$
Associative Properties
$(\boldsymbol{A} \cup \boldsymbol{B}) \cup \boldsymbol{C} = \boldsymbol{A} \cup (\boldsymbol{B} \cup \boldsymbol{C})$ $(\boldsymbol{A} \cap \boldsymbol{B}) \cap \boldsymbol{C} = \boldsymbol{A} \cap (\boldsymbol{B} \cap \boldsymbol{C})$
Distributive Properties
$(\boldsymbol{A} \cup \boldsymbol{B}) \cap \boldsymbol{C} = (\boldsymbol{A} \cap \boldsymbol{C}) \cup (\boldsymbol{B} \cap \boldsymbol{C})$ $(\boldsymbol{A} \cap \boldsymbol{B}) \cup \boldsymbol{C} = (\boldsymbol{A} \cup \boldsymbol{C}) \cap (\boldsymbol{B} \cup \boldsymbol{C})$
Properties of Complement (Taking $\boldsymbol{U}$ as the Whole Universe)
$\neg\oslash = \boldsymbol{U}$ $\neg\boldsymbol{U} = \oslash$ $\boldsymbol{A} \cup \neg\boldsymbol{A} = \boldsymbol{U}$ $\boldsymbol{A} \cap \neg\boldsymbol{A} = \oslash$
De Morgan's Laws
$\neg(\boldsymbol{A} \cup \boldsymbol{B}) = \neg\boldsymbol{A} \cap \neg\boldsymbol{B}$ $\neg(\boldsymbol{A} \cap \boldsymbol{B}) = \neg\boldsymbol{A} \cup \neg\boldsymbol{B}$

$$\mu_{\boldsymbol{A}\cap\boldsymbol{B}}(x) = \mu_{\boldsymbol{A}}(x)\mu_{\boldsymbol{B}}(x) \tag{2}$$

$$x \in (\boldsymbol{A} \cup \boldsymbol{B}) \quad \text{if} \quad x \in \boldsymbol{A} \text{ or } x \in \boldsymbol{B} \tag{3}$$

$$\mu_{\boldsymbol{A}\cup\boldsymbol{B}}(x) = \mu_{\boldsymbol{A}}(x) + \mu_{\boldsymbol{B}}(x) \tag{4}$$

where $\mu_{\boldsymbol{A}}(x)$ represents the degree of the membership of x in a set $\boldsymbol{A}$, i.e., the value of $\mu_{\boldsymbol{A}}(x)$ (that can be 0 or 1) is equal to 0 if x is not member of $\boldsymbol{A}$ and 1 if x is member of $\boldsymbol{A}$. For example, let $\boldsymbol{U}$ and $\boldsymbol{A}$ be defined as follows:

$$\begin{aligned} \boldsymbol{U} &= \{x_1, x_2, x_3, x_4, x_5\} \\ &\boldsymbol{A} = \{x_1, x_3, x_4\} \end{aligned}$$

The values of $\mu_A(x_1)$, $\mu_A(x_3)$ and $\mu_{\boldsymbol{A}}(x_4)$ are equal to 1 if x_1, x_3; and x_4 are members of set A and 0 if not.

Table 2: Intersection Property as a Boolean Product

Intersection Minimum Value		$\mu_A(x)$ 0	1
$\mu_B(x)$	0	0	0
	1	0	1

Table 3: Intersection Property as a Boolean Product

Intersection Maximum Value		$\mu_A(x)$ 0	1
$\mu_B(x)$	0	0	1
	1	1	1

Fuzzy Sets

The notion of *fuzzy sets* was proposed by Lotfi A. Zadeh in 1965 (1). In conventional Boolean algebra, one manipulates the values of membership $\mu_{\mathbf{A}}(x)$ of x in $\boldsymbol{A}$, assuming just the values 0 (zero) or 1 (one). However, the fuzzy set theory, as proposed by Zadeh, assumes all values between 0 and 1 $[0 \leq \mu_{\mathbf{A}}(x) \leq 1]$. In other words, Zadeh's definition of fuzzy sets assumes that $\mu_{\mathbf{A}}(x)$ may assume any value, for example, from a set $\boldsymbol{M} = [0,1]$.

Using this concept, it is possible to introduce the idea of a *linguistic variable* or *fuzzy variable*. A fuzzy variable is a nondeterministic variable that assumes a certain fuzzy subset $\boldsymbol{A}$ defined by its membership function $\mu_{\mathbf{A}}(x)$ instead of numerical values. To describe the proposed concept in mathematical language, one may write

$$\{(x|\mu_{\boldsymbol{A}}(x))\}, \quad \text{for all } x \text{ belonging to } \boldsymbol{U} \tag{5}$$

where $\boldsymbol{U}$ is the universe of discourse. This mathematical expression means: for every x belonging to $\boldsymbol{U}$, it is possible to associate a value of membership $\mu_{\mathbf{A}}(x)$ of x in a defined subset $\boldsymbol{A}$ of $\boldsymbol{U}$.

Therefore, the value of a fuzzy variable is given by a subset of a certain universe of discourse, normally described by words used in day-to-day language. Therefore, the value assumed by a fuzzy variable can also be given by a word in a natural language, such as "small," "big," "tall," "old," "cold," or "hot." Each word is described by a specific subset defined by a particular membership function. Figure 1 shows a graphic description of the membership functions for a fuzzy variable x representing the size of a certain piece of hardware. This fuzzy variable can assume values of "small," "medium," and "big." One can say "the value of x is small," where "the value of x" is the fuzzy variable representing the size and "small" is the fuzzy value. Let the universe of discourse of the variable "size" of a certain piece of hardware be represented by $\boldsymbol{U} = \{0,1,\ldots,10\}$, then the

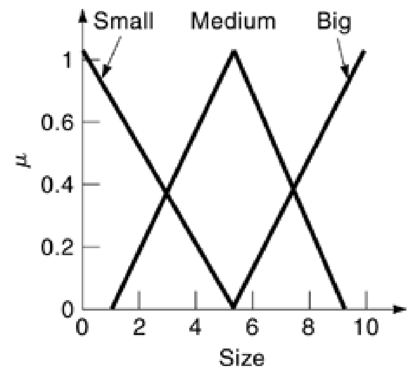


Fig. 1. Fuzzy subsets: small, medium, and big.

membership function of the values of "size" can, for example, be given by

$$\begin{aligned}\text{small} &= \{(0|1), (1|0.8), (2|0.6), (3|0.4), (4|0.2), (5|0).(6|0), (7|0), (8|0), (9|0), (10|0)\}\\ \text{medium} &= \{0|0), (1|0), (2|0.25), (3|0.5), (4|0.75), (5|1), (6|0.75), (7|0.5), (8|0.25), (9|0), (10|0)\}\\ \text{big} &= \{(0|0), (1|0), (2|0), (3|0), (4|0), (5|0), (6|0.2), (7|0.4), (8|0.6), (9|0.8), (10|1)\}\end{aligned}$$

To make these mathematical sentences more understandable, for example, one can take some values of x and, with respect to each membership function ("small," "medium," and "big"); they can be interpreted as follows:

$x = 0$	Belongs to "small" with a degree of membership equal to 1.0
$x = 2$	Belongs to "medium" with a degree of membership equal to 0 or does not belong to "medium"
$x = 7$	Belongs to "big" with a degree of membership equal to 0 or does not belong to "big"

In some cases, the grades of the membership $\mu_{\mathbf{A}}(x)$ can assume values from the set $\boldsymbol{M} = (-\infty, \infty)$ (or $[-1,1]$, for a normalized set). This generalization leads to a more general structure named L–fuzzy sets (2). The letter L comes from the word *lattice* (lattice theory).

As presented for ordinary sets, some properties and basic operations can also be defined for fuzzy sets. In fact, ordinary set theory is a subset of fuzzy set theory. The ordinary set theory uses a set $\boldsymbol{M}_1 = \{0,1\}$ and the fuzzy set theory uses, for example, a set $\boldsymbol{M}_2 = [0,1]$, where $\boldsymbol{M}_1$ is a subset of $\boldsymbol{M}_2$. Therefore, all operations and properties of ordinary set theory are valid for fuzzy set theory. In this way, according to this statement, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are fuzzy subsets, all properties of Table 1 can be used in fuzzy set theory.

In addition, some operations can be redefined. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be fuzzy subsets of a universe of discourse $\boldsymbol{U}$ and x an element of $\boldsymbol{U}$. Table 5 presents some of these basic operations.

For example, using the fuzzy subsets "small" and "medium," as defined above (shown in Fig. 1), to represent the values assumed by a certain fuzzy variable, some computations can be made using these subsets, according to the basic operations presented in Table 5. Table 6 shows these computations.

Basic Concepts of Fuzzy Statements

Data modeling can be defined as an attempt to represent, in a clear way, the available information from a set of data. The data modeling procedure is important because merging data in algebraic expressions allows the

Table 4. Basic Operations

Inclusion
$A \subset B$: $\mu_A(x) \leq \mu_B(x)$ for all x belonging to U
Meaning: A is an enclosure of B or B is an envelope of A
Equality
$A = B$ $\mu_A(x) = \mu_B(x)$ for all x belonging to U
Meaning: A is equal to B
Complement (with Values of $\mu_A(x)$ Given by the Set $M = [0, 1]$)
$\neg A$: $\mu_{\neg A}(x) = 1 - \mu_A(x)$ for all x belonging to U and the values of $\mu_A(x)$ given by the set $M = [0, 1]$
Meaning: $\neg \boldsymbol{A}$ is not $\boldsymbol{A}$
Intersection
$\boldsymbol{A} \cap \boldsymbol{B}$ $\mu_{\boldsymbol{A}\cap\boldsymbol{B}}(x) = \min(\mu_A(x), \mu_B(x))$ for all x belonging to U
Meaning: $\boldsymbol{A}$ and $\boldsymbol{B}$
Union
$\mathbf{A} \cup \mathbf{B}$: $\mu_{A\cup B}(x) = \max(\mu_A(x), \mu_B(x))$ for all x belonging to U
Meaning: $\boldsymbol{A}$ or $\boldsymbol{B}$ (meaning, or $\boldsymbol{A}$ or $\boldsymbol{B}$ or both)
Disjunctive Sum
$\boldsymbol{A} \oplus \boldsymbol{B} = (\boldsymbol{A} \cap \neg \boldsymbol{B}) \cup (\neg \boldsymbol{A} \cap \boldsymbol{B})$
Meaning: $\boldsymbol{A}$ or (exclusive) $\boldsymbol{B}$ (meaning, or $\boldsymbol{A}$ or $\boldsymbol{B}$ but not both)
Difference
$\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} \cap \neg \boldsymbol{B}$
Meaning: $\boldsymbol{A}$ minus $\boldsymbol{B}$
Algebraic Product
$\boldsymbol{A} \bullet \boldsymbol{B}$: $\mu_{\boldsymbol{A}\bullet\boldsymbol{B}}(x) = \mu_{\boldsymbol{A}}(x) \cdot \mu_{\boldsymbol{B}}(x)$ for all x belonging to $\boldsymbol{U}$
Meaning: $\boldsymbol{A}$ and probabilistic $\boldsymbol{B}$
Algebraic Sum
$\boldsymbol{A} \otimes \boldsymbol{B}$: $\mu_{A\otimes B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$ for all x belonging to U
Meaning: A or-probabilistic B

Table 5: Numerical Examples for Basic Operations

Complement

not small = ¬small : $\mu_{\neg\text{small}}(X) = 1 - \mu_{\text{small}}(X)$

not small = {(0|0), (1|0.2), (2|0.4), (3|0.6), (4|0.8), (5|1), (6|1), (7|1), (8|1), (9|1), (10|1)}

Intersection

small and medium = small ∩ medium : $\mu_{\text{small}\cap\text{medium}}(X) = MIN(\mu_{\text{small}}(X), \mu_{\text{medium}}(X))$

small and medium = {(0|0), (1|0), (2|0.25), (3|0.4), (4|0.2), (5|0), (6|0), (7|0), (8|0), (9|0), (10|0)}

Union

small or medium = small ∪ medium : $\mu_{\text{small}\cup\text{medium}}(X) = MAX(\mu_{\text{small}}(X), \mu_{\text{medium}}(X))$

small or medium = {(0|1), (1|0.8), (2|0.6), (3|0.5), (4|0.75), (5|1), (6|0.75), (7|0.5), (8|0.25), (9|0), (10|0)}

Disjunctive Sum

small or (exclusive) medium = small ⊕ medium = (small ∪ ¬medium) ∪(¬small ∩ medium)

small ∩ ¬medium = {(0|1), (1|0.8), (2|0.6), (3|0.4), (4|0.2), (5|0), (6|0), (7|0), (8|0), (9|0), (10|0)}

¬small ∩ medium = {(0|0), (1|0), (2|0.25), (3|0.5), (4|0.75), (5|1), (6|0.75), (7|0.5), (8|0.25), (9|0), (10|0)}

small or (exclusive) medium = {(0|1), (1|0.8), (2|0.6), (3|0.5), (4|0.75), (5|1), (6|0.75), (7|0.5), (8|0.25), (9|0), (10|0)}

Difference

small − medium = small ∩ not medium

small − medium = {(0|1), (1|0.8), (2|0.6), (3|0.4), (4|0.2), (5|0), (6|0), (7|0), (8|0), (9|0), (10|0)}

Algebraic Product

small • medium : $\mu_{\text{small}\bullet\text{medium}}(X) = \mu_{\text{small}}(X) \cdot \text{medium}(X)$

small • medium = {(0|0), (1|0), (2|0.15), (3|0.2), (4|0.15), (5|0), (6|0), (7|0), (8|0), (9|0), (10|0)}

Algebraic Sum

small ⊗ medium : $\mu_{\text{small}\otimes\text{medium}}(X) = \mu_{\text{small}}(X) + \mu_{\text{medium}}(X) - \mu_{\text{small}}(X) \cdot \mu\text{medium}(X)$

small ⊗ medium = {(0|1), (1|0.8), (2|0.7), (3|0.7), (4|0.8), (5|1), (6|0.75), (7|0.5), (8|0.25), (9|0), (10|0)} user to better visualize, understand, and interpret its structure. Moreover, algebraic expressions can be easily handled and incorporated into computer programs.

Fuzzy Statements. As seen before, the degree of membership of an element x of a fuzzy subset $\boldsymbol{A}$ can be denoted by $\mu_A(x)$. To express specific knowledge, input variables can be combined through linguistic conjunctions such as AND and OR, as shown in Table 5. Each linguistic conjunction has a meaning in the fuzzy logic theory and represents a specific operation (AND $\leftrightarrow$ minimum and OR $\leftrightarrow$ maximum). Also, in some cases, the complement of a membership can represent its negation; for example, the complement of $\boldsymbol{A}$ is $\not\boldsymbol{A}$, which means "not $\boldsymbol{A}$." In addition, these concepts allow the use of adverbs to modify (increase or decrease) the sharpness of a linguistic value, such as "very," "quite," and "about." Specific mathematical operations can be related to each adverb according to the desired effect in the shape of the fuzzy subset or linguistic value of the fuzzy variable.

According to these considerations, a fuzzy statement can be defined as an attribution of a fuzzy value to a fuzzy variable. This fuzzy value can be a single value (with or without adverbs) or a composed value (with two or more values that are combined by conjunctions). The general form of a fuzzy statement can be written as

$$X_i \text{ is } \boldsymbol{A}_i \tag{6}$$

where A_i represents the fuzzy value of the fuzzy variable x_i.

A fuzzy statement is a concept possible to be identified in some very simple examples taken from day-to-day life. For example, fuzzy statements can be "Mary is small," "John is tall," "the temperature is hot," and "the value of x is big or not very small," where "Mary," "John," "temperature," and "value of x" are the fuzzy variables of the statements, and "small," "tall," "hot," and "big or not very small" are their fuzzy values, respectively.

Fuzzy Conditional Statements. A mathematical equation represents a mapping between the input and output variable (or variables), and can be represented as a conditional statement in *if—then* form. Several kinds of mapping, such as artificial neural network techniques or linear equations, are described in the literature; they represent a way to manipulate the relations between input and output variable with advantages and drawbacks. Here will be proposed a mapping using fuzzy conditional statements in *if—then* format.

To build a fuzzy conditional statement, an *if—then* rule must have its premise and/or consequence represented by fuzzy statements. A structure based in conditional statement can be interpreted as a decision system: *if* the premise happens *then* the consequence will happen. A typical structure for a decision system has, normally, multiple input and multiple output (*MIMO*) variables. Let us consider a system with p input variables x, in the premise of the rules, and m output variables y, in the consequence. Thus, the general form of a fuzzy conditional statement is as follows:

$$\text{If}\,\{x_1 \text{ is } \boldsymbol{A}_1, \ldots, x_p \text{ is } \boldsymbol{A}_{\mathrm{j}}\}\ \textit{then}\ \{y_1 \text{ is } \boldsymbol{B}_1, \ldots, y_{\mathrm{m}} \text{ is } \boldsymbol{B}_k\} \tag{7}$$

where x_1 to x_p are the input fuzzy variables, y_1 to y_{m} are the output fuzzy variables, and $\boldsymbol{A}_i$ and $\boldsymbol{B}_i$ are the fuzzy values represented by the fuzzy subsets. The premise and the consequence of the fuzzy conditional statements are combined using a comma to denote the conjunction AND.

If the decision system has just one output [multiple input–single output (*MISO*)], the structure for fuzzy conditional statements has the general form given by

$$\text{If}\,\{x_1 \text{ is } \boldsymbol{A}_1, \ldots, x_p \text{ is } \boldsymbol{A}_{\mathrm{j}}\}\ \textit{then}\ \{y_1 \text{ is } \boldsymbol{B}_1\} \qquad (8)$$

Ordinary and Fuzzy Relations

An ordinary relation can be defined as a set of degree of membership of each n-tuple from a set $\boldsymbol{U}_1 \times \boldsymbol{U}_2 \times \cdots \times \boldsymbol{U}_n$ (Cartesian product). For example, let $\boldsymbol{U}_1 = \{a,b,c\}$ and $\boldsymbol{U}_2 = \{d,e,f,g\}$, and $\boldsymbol{M} = [0,1]$. Figure 2 presents a sample of a relation $\Re$, while the following equation shows the numerical values:

$$\begin{aligned} \Re(\boldsymbol{U}_1 \times \boldsymbol{U}_2) \;=\; & \{((a,d)|1), ((a,e)|0), ((a,f)|1), ((a,g)|0), ((b,d)|1), \\ & ((b,e)|1), ((b,f)|1), ((b,g)|0), ((c,d)|0), ((c,e)|0), \\ & ((c,f)|1), ((c,g)|1)\} \end{aligned} \qquad (9)$$

For a fuzzy relation, the definition follows the same structure of ordinary relations and the fuzzy set concepts. With two sets $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ and with x being an element of $\boldsymbol{U}_1$ and y an element of $\boldsymbol{U}_2$, for each element of the set of the ordered pairs (x,y), defined by the Cartesian product $\boldsymbol{U}_1 \times \boldsymbol{U}_2$, there is an associated degree of membership taken in a set $\boldsymbol{M} = [0,1]$. For example, Fig. 3 presents a sample of a relation $\Re$, while Eq. (10) shows the numerical values.

$$\begin{aligned} \Re(\boldsymbol{U}_1 \times \boldsymbol{U}_2) \;=\; & \{((a,d)|1), ((a,e)|0), ((a,f)|0.7), ((a,g)|0.2), \\ & ((b,d)|1), ((b,e)|0.8), ((b,f)|1), ((b,g)|0), \\ & ((c,d)|0.3), ((c,e)|0), ((c,f)|1), ((c,g)|0.8)\} \end{aligned} \qquad (10)$$

In addition, some operations can be redefined. Let $\Re$ and $\Im$ be fuzzy relations defined in $\boldsymbol{U}_1 \times \boldsymbol{U}_2$, and (x,y) an ordered pair of $\boldsymbol{U}_1 \times \boldsymbol{U}_2$. Table 7 presents some of these basic operations using fuzzy relations.

Composition of Two Relations

MinMax Composition between two Relations. Consider two fuzzy (or ordinary) relations $\Re$ and $\Im$ defined in the following Cartesian products $\boldsymbol{X} \times \boldsymbol{Y}$ and $\boldsymbol{Y} \times \boldsymbol{Z}$, respectively. There are many ways to compute another relation $\aleph$, representing the Cartesian product $\boldsymbol{X} \times \boldsymbol{Z}$ based on $\Re$ and $\Im$. Minmax composition (or maxmin composition) is one of these ways.

Using the relations $\Re$ and $\Im$ defined above, the value of each element of $\aleph$ can be computed with

$$\aleph = \Re \circ \Im: \quad \mu_{RoT}(x,z) = \max\,(\,\min(\mu_{\Re}(x,y), \mu_{\Im}(y,z)) \qquad (11)$$

for all (x,y) belonging to $\boldsymbol{X} \times \boldsymbol{Y}$ and for all (y,z) belonging to $\boldsymbol{Y} \times \boldsymbol{Z}$

Table 6. Basic Operations

Inclusion

$\Re \subset \Im$: $\mu_{\Re}(x, y) \le \mu_{\Im}(x, y)$ for all (x, y) belonging to $\boldsymbol{U}_1 \times \boldsymbol{U}_2$

Meaning: $\Re$ is an enclosure of $\Im$ or $\Im$ is an envelope of $\Re$

Equality

$\Re = \Re$: $\mu_{\Re}(x, y) = \mu_{\Im}(x, y)$ for all (x, y) belonging to $\boldsymbol{U}_1 \times \boldsymbol{U}_2$

Meaning: $\Re$ is equal to $\Im$

Complementation

$\neg\Re$: $\mu_{\neg}\Re(x, y) = 1 - \mu_{\Re}(x, y)$ for all (x, y) belonging to $\boldsymbol{U}_1 \times \boldsymbol{U}_2$ and $\boldsymbol{M} = [0, 1]$

Meaning: $\neg\Re$ is not $\Re$

Intersection

$\Re \cap \Im$: $\mu_{\Re\cap\Im}(x, y) = \min(\mu_{\Re}(x, y), \mu_{\Im}(x, y))$ for all (x, y) belonging to $\boldsymbol{U}_1 \times \boldsymbol{U}_2$,

Meaning: $\Re$ and $\Im$

Union

$\Re \cup \Im$: $\mu_{\Re\cup\Im}(x, y) = \max(\mu_{\Re}(x, y), \mu_{\Im}(x, y))$ for all (x, y) belonging to $\boldsymbol{U}_1 \times \boldsymbol{U}_2$,

Meaning: $\Re$ or $\Im$

Disjunctive Sum

$\Re \oplus \Im = (\Re \cap \neg\Im) \cup (\neg\Re \cap \Im)$

Meaning: $\Re$ or (exclusive) $\Im$

Difference

$\Re - \Im = \Re \cap \neg\Im$

Meaning: $\Re$ minus $\Im$

Algebraic product

$\Re \bullet \Im$: $\mu_{\Re\bullet\Im}(x, y) = \mu_{\Re}(x, y) \cdot \mu(x, y)$ for all x belonging to $\boldsymbol{U}_1 \times \boldsymbol{U}_2$,

Meaning: $\Re$ and probabilistic $\Im$

Algebraic sum

$\Re \otimes \Im$: $\mu_{\Re\otimes\Im}(x, y) = \mu_{\Re}(x, y) + \mu_{\Im}(x, y) - \mu_{\Re}(x, y) \cdot \mu_{\Im}(x, y)$ for all (x, y) belonging to $\boldsymbol{U}_1 \times \boldsymbol{U}_2$,

Meaning: $\Re$ or-probabilistic $\Im$

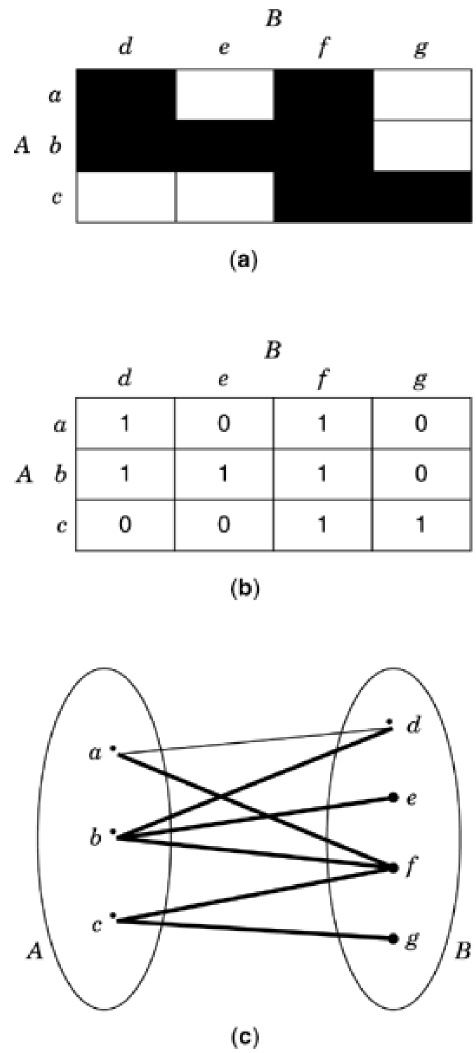
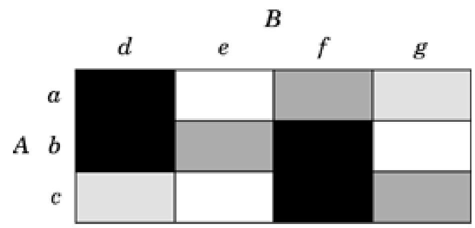


Fig. 2. Relation representations: (a) table, (b) numerical, and (c) graphs.

Numerical example. Let $\Re$ and $\Im$ be the fuzzy relations defined below.

$\Re$	y_1	y_2	y_3	y_4	y_5
x_1	0.1	0.3	0.5	0.7	1
x_2	0.5	0.8	1	1	0.8
x_3	1	0.6	0.2	0	0.2

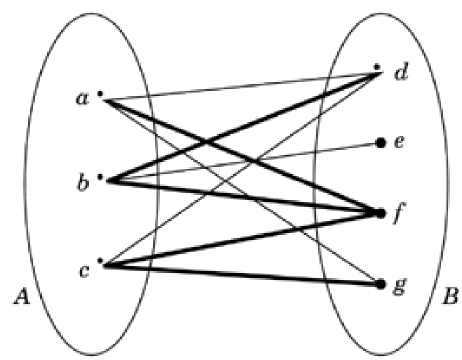
$\Im$	z_1	z_2	z_3	z_4
y_1	0.2	0.4	0	1
y_2	0.7	0.9	1	0.9
y_3	1	0.4	0.2	0.5
y_4	0.3	0.1	0.3	0.3
y_5	0	0.1	0.2	0.1



(a)

		d	e	f	g
	a	1	0	0.7	0.2
A	b	1	0.8	1	0
	c	0.3	0	1	0.8

(b)



(c)

Fig. 3. Relation representations: (a) table, (b) numerical, and (c) graphs.

Computing the value of $\aleph(x_1, z_1)$:

$$\begin{aligned}\aleph\,(x_1, z_1) &= \max\{\min\,(\mu_{\mathfrak{R}}(x_1, y_1), \mu_{\mathfrak{I}}(y_1, z_1)), \min\,(\mu_{\mathfrak{R}}(x_1, y_2), \mu_{\mathfrak{I}}(y_2, z_1)),\\ &\quad \min\,(\mu_{\mathfrak{R}}(x_1, y_3), \mu_{\mathfrak{I}}(y_3, z_1)), \min\,(\mu_{\mathfrak{R}}(x_1, y_4), \mu_{\mathfrak{I}}(y_4, z_1))\}\\ &= \max\{\min(0.1, 0.2), \min(0.3, 0.7), \min(0.5, 1),\\ &\quad \min(0.7, 0.3), \min(1, 0)\}\\ &= \max\{0.1, 0.3, 0.5, 0.3, 0\} = 0.5\end{aligned}$$

Continuing this computation for other elements of $\aleph$, the relation obtained is

$\aleph$	z_1	z_2	z_3	z_4
x_1	0.5	0.4	0.3	0.5
x_2	1	0.4	0.8	0.8
x_3	0.6	0.6	0.6	1

Operations with MinMax Composition and Other Compositions. The minmax composition operation is associative and distributive with respect to union but not with respect to intersection.

$$(\Re \circ \Im) \circ \aleph = \Re \circ (\Im \circ \aleph) \tag{12}$$

$$\Re \circ (\Im \cup \aleph) = (\Re \circ \Im) \cup (\Re \circ \aleph) \tag{13}$$

$$\Re \circ (\Im \cap \aleph) \neq (\Re \circ \Im) \cap (\Re \circ \aleph) \tag{14}$$

There are many other possible compositions like max-product, max-times, and min-product used in some specific conditions. The most natural composition is the minmax, because it is very similar to the matrix product. In this product, there is an algebraic product of each pair and then an algebraic sum of the results. Observing the minmax composition, there is a min composition (in ordinary relations, equal to a Boolean product) of each pair and then a max composition (in ordinary relations, equal to a Boolean sum) of the results.

Relation as Mapping. A relation can also be used as mapping between two worlds, for example $\boldsymbol{X}$ and $\boldsymbol{Y}$. In other words, a fuzzy set $\boldsymbol{A}$ in the first world $\boldsymbol{X}$ has an image (a fuzzy set) $\boldsymbol{B}$ in the world $\boldsymbol{Y}$. Also, there are many ways to compute this image. MinMax is the most frequently used composition.

This mapping can be expressed by the following equation, in which there is a relation (mapping) $\Re$, of a fuzzy set $\boldsymbol{A}$, in $\boldsymbol{X}$ (domain), and a fuzzy set $\boldsymbol{B}$, in $\boldsymbol{Y}$ (range).

$$\mu_{\mathrm{B}}(y) = \max(\min(\mu_{\Re}(x, y), \mu_{\mathrm{A}}(x)) \quad \begin{array}{l}\text{for all}(x, y)\text{belonging to } \boldsymbol{X} \times \boldsymbol{Y}, \\ \text{for all}(x)\text{belonging to } \boldsymbol{X} \\ \text{for all}(y)\text{belonging to } \boldsymbol{Y}\end{array} \tag{15}$$

An example of this computation is carried out below, using the relation $\Re$ defined above and the fuzzy set $\boldsymbol{A}$.

$$\boldsymbol{A} = \{(x_1|1), (x_2|0.8), (x_3|0.6)\}$$

$\Re$	y_1	y_2	y_3	y_4	y_5
x_1	0.1	0.3	0.5	0.7	1
x_2	0.5	0.8	1	1	0.8
x_3	1	0.6	0.2	0	0.2

Computing the value of $\boldsymbol{B}(y_1)$:

$$\begin{aligned}\boldsymbol{B}(y_1) &= \max\{\min(\mu_{\Re}(x_1, y_1), \mu_{\mathrm{A}}(x_1)), \min(\mu_{\Re}(x_2, y_1),\\ &\quad \mu_{\mathfrak{S}}(x_2)), \min(\mu_{\Re}(x_3, y_1), \mu_{\mathfrak{S}}(x_3))\}\\ &= \max\{\min(1, 0.1), \min(0.8, 0.5), \min(0.6, 1)\}\\ &= \max\{0.1, 0.3, 0.6\} = 0.6\end{aligned}$$

Continuing this computation for other elements of $\boldsymbol{B}$, the fuzzy set obtained is

$$\boldsymbol{B} = \{(y_1|0.6), (y_2|0.8), (y_3|0.8), (y_4|0.8)), (y_5|1)\}$$

An Illustrative Example

The Problem. In order to have a more accurate view of the application of MinMax in the fuzzy technique, let us propose a simple and comprehensive control problem. The problem refers to the action of stepping on the brakes of a car in order to stop the car when the driver sees a red light. It is easy to notice that the inputs evaluated by the driver are the speed of the car ($\boldsymbol{v}$) and the distance between the car and the traffic light ($\boldsymbol{d}$).

The idea behind the application of the fuzzy technique is to replace the driver in this action. The human strategy is replaced by a decision-making algorithm described in terms of a set of fuzzy conditional statements.

The analysis of the driver algorithm. When the driver sees the red light, he or she evaluates (measures) the car's speed and the distance to the stop point, and according to the obtained information, he or she decides on the amount of force that has to be exerted on the brake pedal. To translate into words the strategy used by the driver, let us imagine two common situations. The first is when the driver sees the red light and the stopping distance ($\boldsymbol{d}$) is short and the speed of the car ($\boldsymbol{v}$) is high. In this situation, the force applied to the brakes has to be high to avoid disaster. The second situation is when the stopping distance is very short and the speed of the car is very close to zero. Now, the force applied to the brakes can be very small.

The control system takes the input variables, which are numerical values measured directly from the process, and manipulates them using a control algorithm (in this case, the fuzzy decision-making algorithm). The control algorithm generates the output that allows the control process to achieve the desired behavior. These values are not fuzzy, but crisp values. They are presented to the system as continuous values read and manipulated (computed); thus, the algorithm, representing the driver, continuously generates output values to guarantee the best performance of the entire process.

To use a fuzzy technique scheme, the crisp values obtained from the process have to be transformed into fuzzy values; this step is called the *fuzzification* process. The *fuzzification* process is the first step before running the set of conditional fuzzy statements or set of fuzzy rules, which is done after reading the numerical values of variables. The objective of *fuzzification* is to transform the "actual values" of the variables into "fuzzy

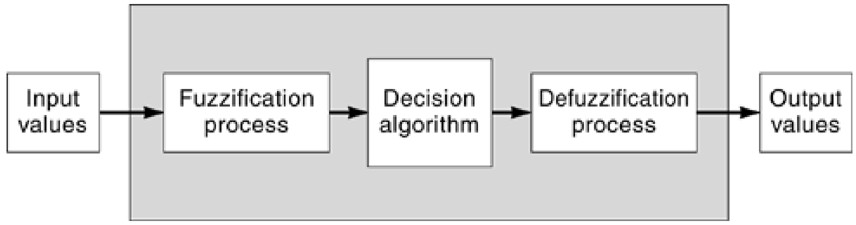


Fig. 4. Scheme for application of the fuzzy technique.

values" (linguistic values), which are then manipulated by the decision algorithm. After running the decision algorithm, the calculated output value is also a fuzzy value that cannot be used as an input for a real process. The *defuzzification* process is the last step of the fuzzy technique in order to transform fuzzy values back to actual values.

Figure 4 presents the general scheme of a decision process using fuzzy technique. Let us define the values that the input fuzzy variables will assume. The fuzzy variable distance from the car to the stopping point ($\boldsymbol{d}$) will assume three fuzzy values: small, medium, and big. The other input fuzzy variable, the speed of the car ($\boldsymbol{v}$), will assume also three fuzzy values: low, medium, and high. And finally, for the output fuzzy variable, force applied to the brake pedal ($\boldsymbol{f}$), also three values will be assigned: low, medium, and high.

Figure 5 illustrates graphically the values assumed for each variable of the process, described by their fuzzy subsets. It is important to note that the fuzzy value medium or high for one variable does not necessarily have to coincide with other variable; in general they are really different subsets.

An example of the fuzzification process is shown in Figure 5. Considering only the first rule depicted in Figure 5, the input $\boldsymbol{d} = 10$ m generates a value of memberships for the subset "small" depending on the premise of the fuzzy rule equal to 0.6 [$\mu_{\text{small}}(\boldsymbol{d}) = \mu_{\text{small}}(10) = 0.6$], and the input $\boldsymbol{v} = 8$ m/s generates a value of membership for the subset "medium," equal to 0.5 [$\mu_{\text{medium}}(\boldsymbol{v}) = \mu_{\text{medium}}(8) = 0.5$]. An example of the *defuzzification* process is presented next.

Fuzzy Inference Process. A fuzzy inference process is the way the set of fuzzy conditional statements (or fuzzy rules) are executed to obtain a meaningful inference in the output. As mentioned before, the complete decision algorithm is composed of a set of fuzzy rules. The algorithm first transforms the input variables into fuzzy statements, and then computes the output value. The execution of each rule is done using *modus ponens*, which means that the premise of each rule produces a degree of membership for a certain value of input variables. This degree of membership is a function of the *fuzzified* values (*fuzzification* procedure) of the input variables and of the conjunctions used among them for each fuzzy rule. Let's consider the example presented in Fig. 5, where the two arbitrary fuzzy conditional statements or fuzzy rules are shown:

$$\text{Rule } i \textit{ If } \boldsymbol{d} \text{ is small } \textbf{and } \boldsymbol{v} \text{ is medium}, \ \textit{then } \boldsymbol{f} \text{ is medium.}$$
$$\text{Rule } j \textit{ If } \boldsymbol{d} \text{ is small } \textbf{and } \boldsymbol{v} \text{ is high}, \ \textit{then } \boldsymbol{f} \text{ is high}$$

Notice that during the *fuzzification* process for rule i, the values obtained for the membership functions are 0.6 and 0.5, as mentioned above. Since the liaison element used between the fuzzy statements is the conjunction "and" (which represents the "minimum" operator), the conclusion of the rule (value "medium") has its value of membership limited by the minimum value of the premise, that is, min(0.6, 0.5) = 0.5. This concept is depicted as a shaded area (S_i) in Fig. 5.

Each rule produces a limited area (as illustrated by the shaded area in Fig. 5) according to the value produced by the premise and the output fuzzy value of the conclusion represented by its membership function.

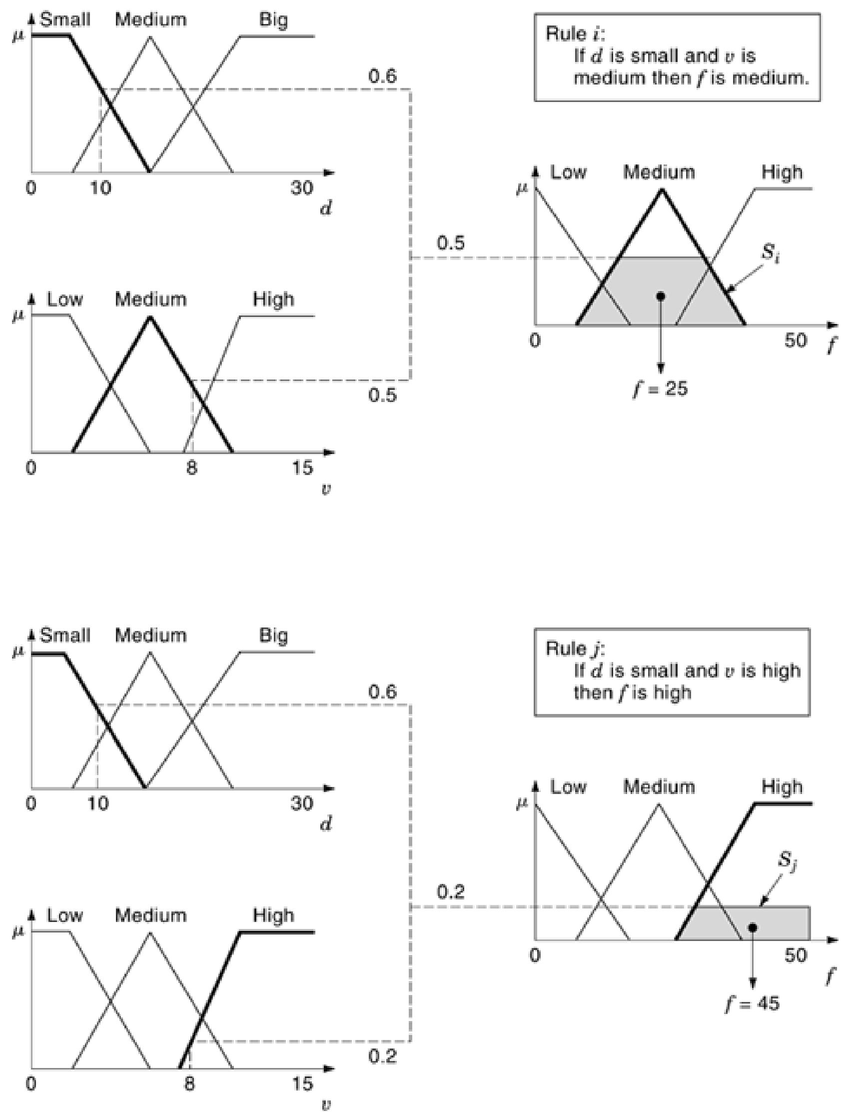


Fig. 5. Fuzzification, inference and defuzzification processes.

In the same way, for rule j, the limitation of the fuzzy value "high" is 0.2. If the value of membership obtained by the premise is zero, it means the rule has no influence on the computation of the final output value.

After the execution of all the rules, the *defuzzification* process begins. The shaded area for each rule is computed; then the maximum operation is applied to define the largest one. In Fig. 5, the shaded area S_i is bigger than S_j, that is, $\max(S_i, S_j) = S_i$. So the area S_i is taken to calculate the final actual output value. This value is computed using the center of gravity method (centroid) for the geometrical figure represented by the shaded area S_i. The value of the abscisa found is the actual value of the output variable. Figure 5 shows an example of the process, where the centroid method produces a value of ***f*** equal to 25 for the largest shaded area obtained (rule S_i). The following describes the centroid method used:

$$f_{(\text{actual output})} = \frac{\sum \mu_{\text{Medium}}(f)f}{\sum \mu_{\text{Medium}}(f)} \tag{16}$$

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