NOMOGRAMS

A nomogram, nomograph, or alignment chart is a graphical display of the relationship between (usually) three variables which allows the value of one of the variables to be determined given the values of the other two variables.

EXAMPLES OF SIMPLE NOMOGRAMS

The underlying concepts of nomograms and the meanings of some fundamental terms can be illustrated by considering some of the basic simple nomogram types and the functional relations treated by them.

The Parallel Nomogram

A simple nomogram for performing the sum $\omega = \mu + u$ is given in Fig. 1. The calibrated axes represent values of the variables μ, u, and ω. A straight line, the *isopleth,* connects the point on the μ-axis corresponding to the value $\mu = 0.4$ with the point on the u-axis corresponding to the value $u =$

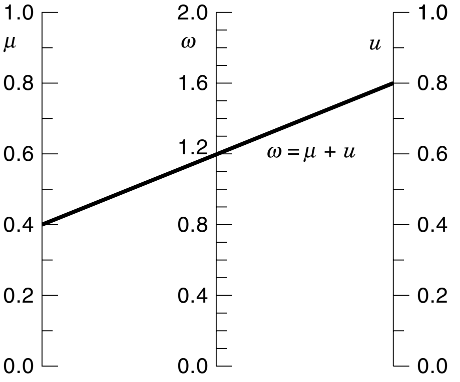

Figure 1. A simple parallel nomogram with linear axes representing the relation $\omega = \mu + u$.

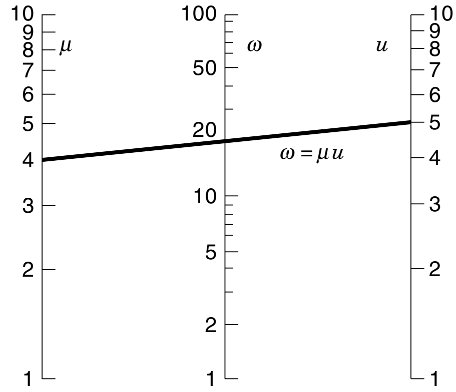

Figure 2. A parallel nomogram with logarithmic axes representing the relation $\omega = \mu u$.

0.8 and intercepts the ω-axis at the value $\omega = 1.2$, the sum of 0.4 and 0.8. In this manner the sum of any pair of μ and u up to a maximum value of 2 can be determined with the corresponding isopleth. The three axes are linear; i.e., changes in the values of a variable correspond to proportional changes in position along the respective axis, the constant of proportionality being the *scale* of the axis. The scale of the ω-axis in this case is half that of the μ- and u-axes. If we were to replace the ω-axis with one of equal scale to that of the μ- and u-axes the nomogram would yield the average of μ and u. By appropriately modifying the scales of the μ- and u-axes or the separation between the parallel axes a nomogram for calculating weighted averages of μ and u can be constructed. By replacing the linear scales in Fig. 1 with appropriate logarithmic scales we can obtain a nomogram which adds logarithms, i.e. which multiplies powers of μ and u (1). A simple nomogram with logarithmic scales for calculating $\omega = \mu u$ is illustrated in Fig. 2.

The Z Nomogram

A Z nomogram with linear scales is illustrated in Fig. 3. In this example the percent of students passing an examination (% pass) is calculated from the total number of students passing and the total number of students failing the examination. For example, 20 passing and 5 failing produces an 80% pass rate. This nomogram expresses the relation

$$\% \text{ Passing} = 100 \frac{\text{Passing}}{\text{Passing} + \text{Failing}} \tag{1}$$

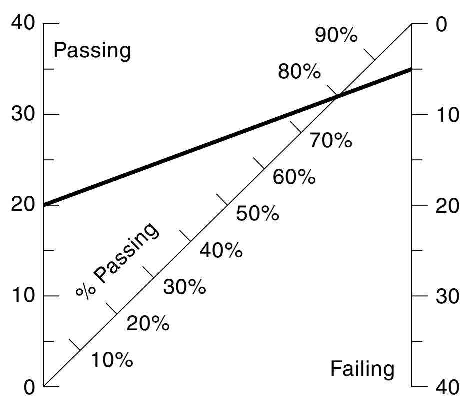

Figure 3. A Z nomogram with linear scales which can be used to calculate the percent pass rate from the total number passing and total number failing.

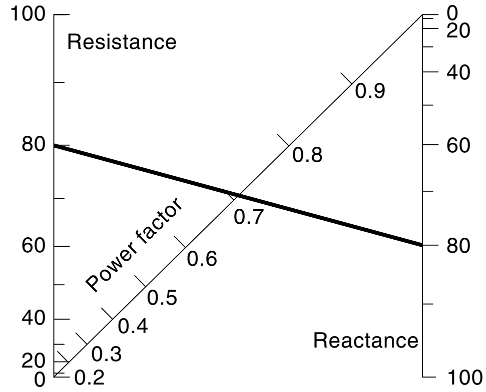

Figure 4. A Z nomogram with parabolic scales which can be used to calculate the power factor from the resistive and reactive components of load impedance.

In general a Z nomogram with linear scales expresses a relation of the form

$$\omega = \frac{\mu}{\mu + u} \tag{2}$$

or any relation which can be cast into this form with a suitable transformation of variables and corresponding nonlinear scales. For example, if a load impedance has a resistive component of R and a reactive component of X, then the power factor can be expressed as

$$(\text{Powerfactor})^2 = \frac{R^2}{R^2 + X^2} \tag{3}$$

Comparing Eq. (3) with Eq. (2) we have the correspondence

$$\begin{aligned} (\text{Powerfactor})^2 &= \omega \\ R^2 &= \mu \\ X^2 &= u \end{aligned} \tag{4}$$

which defines the appropriate nonlinear scales. A nomogram constructed using Eq. (4) is given in Fig. 4.

The W Nomogram

A W chart nomogram with linear scales is illustrated in Fig. 5. In this example the nomogram determines the electrical resistance R resulting from two resistances R_1 and R_2 in parallel. The nomogram performs the calculation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \tag{5}$$

which illustrates the basic functional form treated by this type of nomogram.

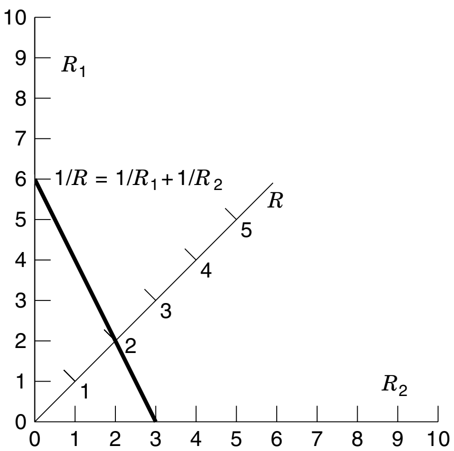

Figure 5. A W nomogram with linear axes for calculating the resultant resistance R of two resistances R_1 and R_2 in parallel.

The Circular Nomogram

A basic circular nomogram is illustrated in Fig. 6. The upper and lower semicircles are two axes with linear scales for the variables α and β, respectively, which for simplicity are also the angles between the respective radii and the x-axis. In this case the radius of the circle is one so that x runs from -1 to $+1$. The values of α, β, and x related by the isopleth satisfy

$$\frac{1-x}{1+x} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2} \tag{6}$$

which illustrates the basic functional form treated by this type of nomogram. This nomogram is typically employed in quality assurance (2). Suppose a small sample of n light bulbs is selected at random from an assembly line and tested until the first failure is recorded at H hours of continuous use. The probability (W) that in a large sample of bulbs no more than $K\%$ of failures will be observed after H hours is given by

$$W = 1 - (1 - 0.01K\%)^n \tag{7}$$

This formula may be rewritten as

$$-\frac{\ln(1-W)}{3} = \frac{n}{30}(-10\ln(1-0.01K\%)) \tag{8}$$

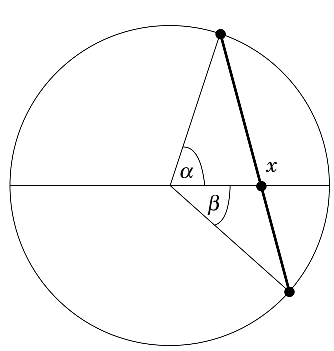

Figure 6. A circular nomogram which shows relation between angles (α, β) subtended by points on upper and lower semicircular axes to center, the x-intercept, and the isopleth.

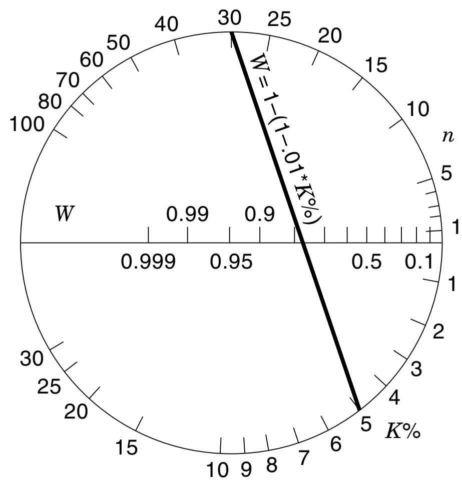

Figure 7. If in a sample of 30 light bulbs the first failure is observed after H hours, then there is a 0.79 probability that in a large sample of bulbs less than 5% will fail in H hours.

Comparing Eq. (8) with Eq. (6) we can make the correspondence

$$\begin{aligned} -\frac{\ln(1-W)}{3} &= \frac{1-x}{1+x} \\ \frac{n}{30} &= \tan\frac{\alpha}{2} \\ -10\ln(1-0.01K\%) &= \tan\frac{\beta}{2} \end{aligned} \tag{9}$$

Inverting these expressions we have

$$\begin{aligned} x &= \frac{2}{1-\dfrac{\ln(1-W)}{3}} - 1 \\ \alpha &= 2\tan^{-1}\frac{n}{30} \\ \beta &= 2\tan^{-1}(-10\ln(1-0.01K\%)) \end{aligned} \tag{10}$$

which can be used to calibrate the α-, β-, and x-axis in terms of n, $K\%$, and W respectively. The resulting nomogram is given in Fig. 7.

GENERAL THEORY OF NOMOGRAMS

We now present the detailed theory of the nomogram treating explicitly nomograms consisting of parallel scales as in Figs. 1 and 2 for expository convenience. Nomograms of this type represent functional relationships of the form

$$f_1(\mu) + f_2(u) = f_3(\omega) \tag{11}$$

We refer the axes of the nomograms to a Cartesian coordinate systems where the μ-, u-, and ω-axes are parallel to the y-axis at x values 0, b, and a, respectively, so that a given value of μ corresponds to the coordinates

$$(X_1, Y_1) = (0, m_\mu f_1(\mu)) \tag{12}$$

where m_μ is a scale factor which associates the value of the function $f_1(\mu)$ with a position on the μ-axis. Likewise, given values of u and ω will correspond to coordinates

$$(X_2, Y_2) = (b, m_u f_2(u)) \tag{13}$$

and

$$(X_3, Y_3) = (a, m_\omega f_3(\omega)) \tag{14}$$

respectively. When these three points are collinear, i.e. fall along an isopleth, their coordinates satisfy the homogeneous determinant relation

$$\begin{vmatrix} X_1 & Y_1 & 1 \\ X_2 & Y_2 & 1 \\ X_3 & Y_3 & 1 \end{vmatrix} = 0 \tag{15}$$

The Construction Determinant

Substituting Eqs. (12), (13), and (14) into Eq. (15) we have

$$\begin{vmatrix} 0 & m_\mu f_1(\mu) & 1 \\ b & m_u f_2(u) & 1 \\ a & m_\omega f_3(\omega) & 1 \end{vmatrix} = 0 \tag{16}$$

Eq. (16) is referred to as the construction determinant for the nomogram (3). Expanding the determinant and rearranging terms we have

$$\left(1 - \frac{a}{b}\right) \frac{m_\mu}{m_\omega} f_1(\mu) + \frac{a}{b} \frac{m_u}{m_\omega} f_2(u) = f_3(\omega) \tag{17}$$

Comparing Eq. (17) with Eq. (11) we have the correspondence

$$\left(1 - \frac{a}{b}\right) \frac{m_\mu}{m_\omega} = 1 \tag{18}$$

and

$$\frac{a}{b} \frac{m_u}{m_\omega} = 1 \tag{19}$$

which are the constraints between the scale factors of the axes and the distances between the axes. Suppose we wish to construct a nomogram to add two variables in quadrature, e.g.

$$\mu^2 + u^2 = \omega^2 \tag{20}$$

where μ runs from 0 to 4 and u from 0 to 5 and that the size of the nomogram is to be 5 centimeters high by 3 centimeters wide. The scale factors for the μ- and u-axes are determined by requiring the range of values to fit the 5-cm height of the nomogram:

$$m_\mu = \frac{5}{4^2} = \frac{5}{16}, \quad m_u = \frac{5}{5^2} = \frac{1}{5}$$

and the separation between the μ- and u-axes by the width of the nomogram:

$$b = 3$$

Dividing Eqs. (18) and (19) and substituting the values m_μ, m_u, and b, allow a to be determined as

$$a = \frac{b}{\dfrac{m_u}{m_\mu} + 1} = \frac{75}{41}$$

Finally, m_ω may be calculated as

$$m_\omega = \frac{a}{b} m_u = \frac{5}{41}$$

Thus, the calibrated scales of the nomogram are calculated as

$$\begin{aligned} X_1 &= 0 & Y_1 &= \frac{5}{16}\mu^2 \\ X_2 &= 3 & Y_2 &= \frac{1}{5}u^2 \\ X_3 &= \frac{75}{41} & Y_3 &= \frac{5}{41}\omega^2 \end{aligned} \tag{21}$$

The nomogram is given in Fig. 8.

Shifting Axis Origins

In this nomogram the origin of the axes correspond to the 0 values of the respective variables. In many situations it is preferable to have the axis origins corresponding to non-0 initial values; e.g., μ_0, u_0, and ω_0 which satisfy Eq. (11). In this case the construction determinant is modified as

$$\begin{vmatrix} 0 & m_\mu[f_1(\mu) - f_1(\mu_0)] & 1 \\ b & m_u[f_2(u) - f_2(u_0)] & 1 \\ a & m_\omega[f_3(\omega) - f_3(\omega_0)] & 1 \end{vmatrix} = 0 \tag{22}$$

and Eq. (18) and Eq. (19) are still valid. If for example $\mu_0 = 1$ and $u_0 = 2$ the scale factors and design parameters become

$$m_\mu = \frac{5}{5^2 - 1^2} \qquad m_u = \frac{5}{6^2 - 2^2} \qquad a = \frac{12}{7}$$

and

$$m_\omega = \frac{5}{56}$$

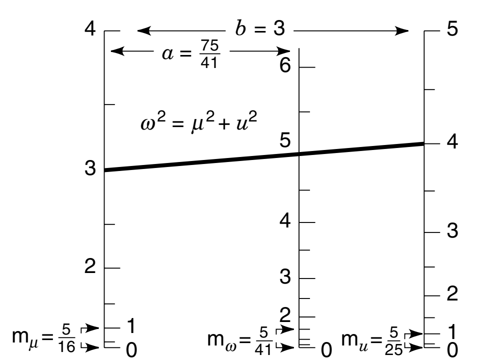

Figure 8. Parallel nomogram for adding variables in quadrature. The nomogram is 5 units high by 3 units wide and illustrates design parameters a, b, and m_μ, m_u, and m_ω.

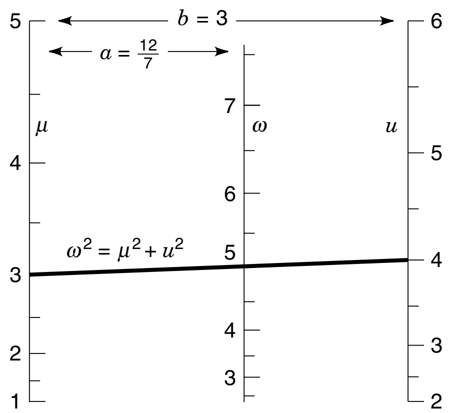

Figure 9. Nomogram as in Fig. 8 with shifted origins.

The calibrated scales for the nomogram are calculated as

$$\begin{aligned} X_1 &= 0 \qquad Y_1 = \frac{5}{24}(\mu^2 - 1) \\ X_2 &= 3 \qquad Y_2 = \frac{5}{32}(u^2 - 4) \\ X_3 &= \frac{12}{7} \qquad Y_3 = \frac{5}{56}(\omega^2 - 5) \end{aligned} \tag{23}$$

This nomogram is given in Fig. 9.

Other Common Nomograms

We now apply the construction determinant methodology to the other nomogram types treated thus far, justifying the functional forms consistent with three nomograms. As stated earlier the Z chart nomogram can be used to represent the functional relationship from

$$f_3(\omega) = \frac{f_1(\mu)}{f_1(\mu) + f_2(u)} \tag{24}$$

Referring to Fig. 10 we can assign the following coordinates to the three points on the isopleth

$$\begin{aligned} X_1 &= 0 & Y_1 &= m_\mu f_1(\mu) \\ X_2 &= a & Y_2 &= Y_{\max} - m_u f_2(u) \\ X_3 &= \frac{a}{[a^2 + Y_{\max}^2]^{1/2}} m_\omega f_3(\omega) & Y_3 &= \frac{Y_{\max}}{[a^2 + Y_{\max}^2]^{1/2}} m_\omega f_3(\omega) \end{aligned} \tag{25}$$

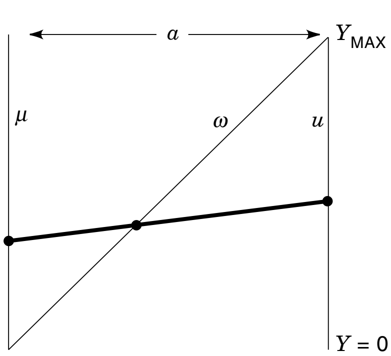

Figure 10. Prototype of Z nomogram illustrating parameters a and $Y_{\max}$.

Where a is the separation between the parallel axes which are both of length $Y_{\max}$. Evaluating the construction determinant with the substitutions of Eqs. (25) leads to

$$f_3(\omega) = [a^2 + Y_{\max}^2]^{1/2} \frac{m_\mu}{m_\omega} \frac{f_1(\mu)}{m_\mu f_1(\mu) + m_u f_2(u)} \tag{26}$$

which is equivalent to Eq. (24) when

$$\frac{m_\omega}{[a^2 + Y_{\max}^2]^{1/2}} = 1 \tag{27}$$

and

$$m_\mu = m_u \tag{28}$$

Equation (27) allows the scale factor m_ω to be determined from the dimensions of the nomogram. Scale factors m_μ and m_u are equal and determined by the length of the nomogram and the range of values of μ and u as in the parallel nomogram treated previously.

The W chart nomogram can be used to represent functional relationship of the form

$$\frac{1}{f_3(\omega)} = \frac{1}{f_1(\mu)} + \frac{1}{f_2(u)} \tag{29}$$

Referring to Fig. 11 we can make the following assignments for the coordinates of the three points on the isopleth

$$\begin{aligned} X_1 &= m_\mu (f_1(\mu)) & Y_1 &= 0 \\ X_2 &= \cos\beta\, m_u f_2(u) & Y_2 &= \sin\beta\, m_u f_2(u) \\ X_3 &= \cos\alpha\, m_\omega f_3(\omega) & Y_3 &= \sin\alpha\, m_\omega f_3(\omega) \end{aligned} \tag{30}$$

Evaluating the construction determinant with these substitutions leads to

$$\frac{1}{f_3(\omega)} = \frac{m_\omega}{m_\mu} \frac{\sin(\beta - \alpha)}{\sin\beta} \frac{1}{f_1(\mu)} + \frac{m_\omega}{m_u} \frac{\sin\alpha}{\sin\beta} \frac{1}{f_2(u)} \tag{31}$$

Comparing Eq. (31) with Eq. (29) we have

$$\frac{m_\omega}{m_\mu} \frac{\sin(\beta - \alpha)}{\sin\beta} = 1 \tag{32}$$

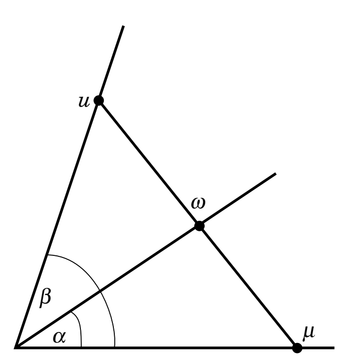

Figure 11. Prototype of W nomogram illustrating parameters α and β.

and

$$\frac{m_\omega}{m_u}\frac{\sin\alpha}{\sin\beta} = 1 \tag{33}$$

The scale factors and angles are determined by the range of the variables, the size of the nomogram, and the constraint of Eq. (32) and Eq. (33). For example, in Fig. 5 the R_1- and R_2-axes (i.e., the μ- and u-axes) have equal scale (e.g. $m_\mu = m_u = 1$) and $\beta = 90°$. Dividing Eq. (32) by Eq. (33) gives

$$\frac{\sin(\beta-\alpha)}{\sin(\alpha)} = 1 \tag{34}$$

which requires $\alpha = 45°$. Substituting into Eq. (33) we have $m_\mu = \sqrt{2}$.

In the simple circular nomogram given in Fig. 6 the X and Y coordinates of the three points of the isopleth are

$$\begin{aligned} X_1 &= \cos\alpha \quad & Y_1 &= \sin\alpha \\ X_2 &= \cos\beta \quad & Y_2 &= -\sin\beta \\ X_3 &= x \quad & Y_3 &= 0 \end{aligned} \tag{35}$$

Substituting these into Eq. (15) and rearranging terms we arrive at

$$\frac{1-x}{1+x} = \tan\frac{\alpha}{2}\tan\frac{\beta}{2} \tag{36}$$

Functional relationships of the form

$$f_3(\omega) = f_1(\mu)f_2(u) \tag{37}$$

may be represented by circular nomograms using the scale transformation

$$\begin{aligned} \alpha &= 2\tan^{-1}(m_\mu f_1(\mu)) \\ \beta &= 2\tan^{-1}(m_u f_2(u)) \\ x &= \frac{2}{m_\mu m_u f_3(\omega)+1} - 1 \end{aligned} \tag{38}$$

DESIGN STRATEGY

Up to this point the construction determinant has been used to justify the functional relation consistent with a given nomogram topology (i.e. parallel, Z, W, circular). We now illustrate the strategy for using the construction determinant to determine the nomogram topology given a nonstandard functional relation, for example

$$f_1(\mu) + f_2(u)f_3(\omega) - f_3^2(\omega) = 0 \tag{39}$$

The first step is to find a 3 × 3 matrix whose determinant is the left hand side of Eq. (39). This can be done by introducing the dummy variables

$$x = f_1 \qquad y = f_2 \qquad z = 1$$

so that Eq. (39) can be thought of as the result of eliminating these variables from the three equations

$$\begin{aligned} x - f_1 z &= 0 \\ y - f_2 z &= 0 \\ x + f_3 y - f_3^2 z &= 0 \end{aligned} \tag{40}$$

Equation (40) can be written in matrix form as

$$\begin{bmatrix} 1 & 0 & -f_1 \\ 0 & 1 & -f_2 \\ 1 & f_3 & -f_3^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \tag{41}$$

The consistency of Eq. (40) requires

$$\det \begin{bmatrix} 1 & 0 & -f_1 \\ 0 & 1 & -f_2 \\ 1 & f_3 & -f_3^2 \end{bmatrix} = 0 \tag{42}$$

which leads directly to Eq. (39). The second step is to use standard determinant identities to transform the determinant of Eq. (42) to the form of the construction determinant, Eq. (15). We proceed by replacing column one with the sum of columns one and two resulting in

$$\begin{vmatrix} 1 & 0 & f_1 \\ 1 & 1 & f_2 \\ 1+f_3 & f_3 & f_3^2 \end{vmatrix} = 0 \tag{43}$$

where we have also multiplied column 3 by −1. Dividing row 3 by $1 + f_3$ and rearranging columns we obtain

$$\begin{vmatrix} 0 & f_1 & 1 \\ 1 & f_2 & 1 \\ \dfrac{f_3}{1+f_3} & \dfrac{f_3^2}{1+f_3} & 1 \end{vmatrix} = 0 \tag{44}$$

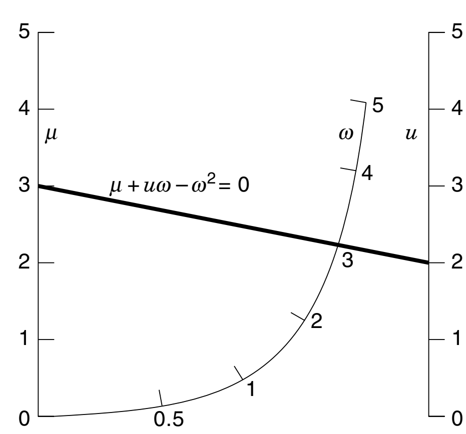

Figure 12. Nonstandard nomogram, with curved axis, generated using the construction determinant.

which is in the form of a construction determinant. The coordinates of the required axes are therefore:

$$\begin{aligned} X_1 &= 0 & Y_1 &= f_1 \\ X_2 &= 1 & Y_2 &= f_2 \\ X_3 &= \frac{f_3}{1+f_3} & Y_3 &= \frac{f_3^2}{1+f_3} \end{aligned} \tag{45}$$

In the simple case where

$$f_1 = \mu, \quad f_2 = u, \quad f_3 = \omega \tag{46}$$

we have the nomogram given in Fig. 12.

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NOMOGRAPH. See NOMOGRAMS.
NONCONTACT THERMOMETERS. See PYROMETERS.