NONLINEAR EQUATIONS

In this article a decomposition method is presented for solving nonlinear equations arising in the study of radiative transfer such as the Chandrasekhar H-equation, conservative hyperbolic systems and nonlinear waves including the Korteweg–de Vries (KdV), and the Klein–Gordon equations. An essential feature of this numerical technique is its rapid convergence and the high degree of accuracy by which it approximates a solution using only a few terms of its iterative scheme. Other techniques including perturbation methods, finite element, finite difference and Galerkin approximation have been used in the literature to solve nonlinear equations (21–33).

This article presents a decomposition technique for solving nonlinear equations. The original method was first introduced by Adomain (1). Numerous articles have been written using this method to solve partial, ordinary and delay differential equations, nonlinear algebraic equations, and boundary-value problems (1–17). The scheme assumes an infinite solution

$$u = \sum_{n=0}^{\infty} u_n$$

where the terms u_n are recursively determined. A common feature to note of problems solved by the decomposition method is that the solution of the underlying equations obtained by this method approximates the exact solution with a high degree of accuracy using only a few terms of the iterative scheme. A modified version of the technique will be presented to handle some of the nonlinear equations we will be dealing with. Four basic equations that are of importance in mathematical physics and engineering will be considered. In particular, we will present the well-known *H-equation* due to Chandrasekhar (18) which arises in the study of radiative transfer; two nonlinear wave equations: the KdV equation, that arises in the modelling of shallow water waves and the Klein–Gordon equation which is an important model in quantum mechanics. Finally, the method will be implemented for solving a hyperbolic conservative system that models shocks.

In the sections that follow we will present the decomposition method along with the results on convergence, the H-equation, the KdV equation, the hyperbolic conservative system, as well as the Klein–Gordon equation.

DECOMPOSITION METHOD

Recently, there has been a great deal of interest (1–12) in applying the Adomian decomposition technique for solving a wide class of nonlinear equations including algebraic, differential, partial-differential, differential-delay and integro-differential equations. The main thrust of this technique is that the solution which is expressed as an infinite series converges very fast to exact solutions. In (19,20) a proof of convergence of the method has been given by employing fixed point theorems. Most recently, Cherruault et al. (19) presented new proofs of convergence with less stringent hypotheses that are more adaptable to dealing with physical problems. A theoretical analysis for the method has been discussed in (15).

In general, we seek a solution to the following nonlinear equation

$$u = L(u) + N(u) + g \tag{1}$$

where L is a linear operator, N is a nonlinear operator and g is a known function in the underlying function space which is normally a Hilbert space. The decomposition technique consists of representing the solution as an infinite series, namely,

$$u = \sum_{n=0}^{\infty} u_n \tag{2}$$

where the terms u_n are to be recursively computed. Also the nonlinear operator N is decomposed as follows:

$$N(u) = \sum_{n=0}^{\infty} A_n \tag{3}$$

where $A_n = A_n(u_0, u_1, u_2, \ldots, u_n)$ are the so-called *Adomian polynomials.* Substituting Eqs. (2) and (3) into Eq. (1) yields

$$\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} L(u_n) + \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n) + g \tag{4}$$

Assuming convergence of the series in Eq. (4), both sides of Eq. (4) will match by setting

$$\begin{cases} u_0 = g \\ u_1 = L(u_0) + A_0(u_0) \\ u_2 = L(u_1) + A_1(u_0, u_1) \\ \vdots \\ u_{n+1} = L(u_n) + A_n(u_0, u_1, \ldots, u_n) \\ \vdots \end{cases} \tag{5}$$

Thus, from Eq. (5) the $u_n's$ given in Eq. (2) can be obtained in a recurrent manner and hence u is determined.

There are important questions to raise now:

1. How are the Adomian polynomials A_n determined?
2. Do the series in Eqs. (2) and (3) always converge? If so, to which function do they converge?

Before we proceed, we give a heuristic argument for determining A_n's when $N(u) = f(u)$ and $f(u)$ is a scalar function.

The Taylor expansion of $f(u)$ around u_0 is:

$$f(u) = f(u_0) + f^{(1)}(u_0)(u - u_0) + \frac{1}{2!} f^{(2)}(u_0)(u - u_0)^2 + \ldots \tag{6}$$

If u is given as an infinite sum

$$u = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \ldots \tag{7}$$

then upon substituting the difference $u - u_0$ from Eq. (7) into Eq. (6), we get

$$\begin{aligned} f(u) = f(u_0) &+ f^{(1)}(u_0)(u_1 + u_2 + \ldots) \\ &+ \frac{1}{2!} f^{(2)}(u_0)(u_1 + u_2 + \ldots)^2 + \ldots \end{aligned} \tag{8}$$

and when simplified this results in:

$$\begin{aligned} f(u) = f(u_0) &+ f^{(1)}(u_0)(u_1 + u_2 + u_3 + \ldots) \\ &+ \frac{1}{2!} f^{(2)}(u_0)(u_1^2 + 2u_1u_2 + 2u_1u_3 + u_2^2 + 2u_2u_3 + u_3^2 + \ldots) \\ &+ \frac{1}{3!} f^{(3)}(u_0)(u_1^3 + 3u_1^2 + 3u_1^2u_2 + 3u_1^2u_3 + 3u_1u_2^2 + \ldots) \\ &+ \ldots \end{aligned} \tag{9}$$

Adomian polynomials are obtained by a reordering and rearranging of the terms given in Eq. (9). Indeed, to determine the Adomian polynomials, one needs to determine the order of each term in Eq. (9) which actually depends on both the subscripts and the exponents of the u_n's. For example, u_1 is of order 1; u_1^2 is of order 2; u_2^3 is of order 6; and so on. If a particular term involves the multiplication of u_n's, its order is determined by the sum of the orders of the u_n's in the term. For example, $u_2^3 u_1^2$ is of order 8 since (3)(2) + (2)(1) = 8. Therefore, rearranging the terms in the expansion Eq. (9) according to the order and assuming that $N(u)$ is as given in Eq. (3) will give the A_n as

$$\begin{cases} A_0 = f(u_0) \\ A_1 = u_1 f^{(1)}(u_0) \\ A_2 = u_2 f^{(1)}(u_0) + \dfrac{1}{2!} u_1^2 f^{(2)}(u_0) \\ A_3 = u_3 f^{(1)}(u_0) + u_1 u_2 f^{(2)}(u_0) + \dfrac{1}{3!} u_1^3 f^{(3)}(u_0) \\ \vdots \end{cases} \tag{10}$$

We will now briefly present the general method and refer the reader to (15) for a more detailed study. As was pointed out earlier the Adomian algorithm assumes a series solution for u given by Eq. (2) and that the nonlinear operator $N(u)$ can be decomposed into:

$$N(u) = \sum_{n=0}^{\infty} A_n \tag{11}$$

The Adomian polynomials A_n's are given by the general formula

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N\left(\sum \lambda^i u_i \right) \right]_{\lambda=0} \qquad n = 0, 1, \ldots \tag{12}$$

Once the A_n are determined by Eq. (12), one can recurrently determine the terms u_n of the series and hence the solution u of Eq. (1). The convergence of the series solution has been established (15,19). The two hypotheses that are necessary for proving convergence of Adomian technique are given in (19) by:

1. The nonlinear functional Eq. (1) has a series solution $\sum_{n=0}^{\infty} u_n$ such that $\sum_{n=0}^{\infty} (1 + \epsilon)^n |u_n| < \infty$ where $\epsilon > 0$ may be very small.
2. The nonlinear operator $N(u)$ is analytic and can be developed in series according to u: $N(u) = \sum_{n=0}^{\infty} \alpha_n u^n$

These conditions are generally satisfied in the modeling of many physical problems.

To illustrate the scheme, let $N(u)$ be a nonlinear function of u, say $f(u)$, where

$$u = u_0 + \lambda u_1 + \lambda^2 u_2 + \ldots$$

then the first four Adomian's polynomials A_n are given by

$$\begin{cases} A_0 = f(u(\lambda))|_{\lambda=0} = f(u_0) \\ A_1 = (df/du)(du/d\lambda)|_{\lambda=0} \\ A_2 = \frac{1}{2}[(d^2g/du^2)(du/d\lambda)^2 + (df/du)(d^2u/d\lambda^2)]|_{\lambda=0} \\ A_3 = \frac{1}{6}[(d^3f/du^3)(du/d\lambda)^3 + 2(d^2f/du^2)(du/d\lambda)(d^2u/d\lambda^2) \\ \quad + (d^2f/du^2)(d^2u/d\lambda^2)(du/d\lambda) + (df/du)(d^3u/d\lambda^3)]|_{\lambda=0} \\ \vdots \end{cases} \tag{13}$$

The $A_n's$ can finally be written in the following convenient way

$$A_n = \sum_{v=1}^{n} c(v,n) f^{(v)}(u_0) \tag{14}$$

This results in the polynomials given in Eq. (10).

In the next sections, this method and a modified version of it will be used for solving several interesting nonlinear equations which are of physical importance. We will begin with the Chandrasekhar equation (18).

CHANDRASEKHAR *H*-EQUATION

In this section, the decomposition method is applied to the Chandrasekhar H-equation given by:

$$H(x) = 1 + H(x)\int_0^1 \frac{x}{x+t}\Psi(t)H(t)dt \tag{15}$$

where the H-function, $H(x)$, measures the emergent radiation and $\Psi(t)$ is referred to as the characteristic function and is a measure of phase. This equation arises in the formulation of problems in the theory of radiative transfer in semi-infinite atmospheres. *Radiative transfer* is the angular distribution of the emergent radiations which results from scattering. For standard problems in isotropic scattering, these angular distributions of the emergent radiations are directly expressed in terms of the *H-functions.*

In Eq. (15), the function $\Psi(t)$ is usually a nonnegative even polynomial in t satisfying

$$\int_0^1 \Psi(t)dt \le \frac{1}{2}$$

It is well known that a positive and continuous solution of Eq. (15) exists (18).

A case to consider, is when the law of diffuse reflection of scattering is given in terms of the phase function $\Psi_0(1 + x\cos\theta)$. This can be expressed in terms of the H-equation corresponding to the following particular choice of the *characteristic function,* $\Psi(t)$ (18)

$$\Psi(t) = \frac{1}{2}\Psi_0[1 + x(1-\Psi_0)t^2] \tag{16}$$

where Ψ_0 is a constant.

If we set

$$z(x) = \frac{1}{H(x)} \tag{17}$$

then Eq. (15) can be written as

$$z(x) = 1 - x\int_0^1 \frac{\Psi(t)}{x+t}\frac{1}{z(t)}dt \tag{18}$$

The nonlinear term in Eq. (18) is,

$$N(z) = f(z) = \frac{1}{z} \tag{19}$$

Thus, upon writing $z(x) = \sum_{n=0}^{\infty} z_n(x)$, and $N(z(x)) = 1/z(x) = \sum_{n=0}^{\infty} A_n(z(x))$ in terms of the Adomian polynomials, Eq. (18) becomes,

$$\sum_{n=0}^{\infty} z_n(x) = 1 - x\sum_{n=0}^{\infty}\int_0^1 \frac{\Psi(t)}{x+t}A_n(t)dt \tag{20}$$

Applying the decomposition method to Eq. (20), the various iterates are given by

$$z_0 = 1 \tag{21}$$

and

$$z_{n+1}(x) = -x\int_0^1 \frac{\Psi(t)}{x+t}A_n(t)dt, \quad n \ge 1 \tag{22}$$

where, upon using Eq. (10), the Adomian polynomials for the nonlinear operator given in Eq. (19) are:

$$\begin{cases} A_0 = \frac{1}{z_0} \\ A_1 = -\frac{1}{z_0^2}z_1 \\ A_2 = -\frac{1}{z_0^2}z_2 + \frac{1}{z_0^3}z_1^2 \\ A_3 = -\frac{1}{z_0^2}z_3 + \frac{2}{z_0^3}z_1z_2 - \frac{1}{z_0^4}z_1^3 \\ \vdots \end{cases} \tag{23}$$

Substituting Eq. (21) into Eq. (23) gives:

$$\begin{cases} A_0 = 1 \\ A_1 = -z_1 \\ A_2 = z_1^2 - z_2 \\ A_3 = -z_1^3 + 2z_1z_2 - z_3 \end{cases} \tag{24}$$

From Eqs. (24), (21), and (22) the first few iterates are:

$$\begin{cases} z_0(x) = 1 \\ z_1(x) = -x\int_0^1 \frac{\Psi(t)}{x+t}dt \\ z_2(x) = x\int_0^1 \frac{\Psi(t)}{x+t}z_1(t)dt \\ z_3(x) = -x\int_0^1 \frac{\Psi(t)}{x+t}(z_1^2(t) - z_2(t))dt \\ \vdots \end{cases} \tag{25}$$

Evaluating the integrals in Eq. (25) (using the computer algebra system maple) yields

$$\begin{cases} z_0(x) = 1 \\ z_1(x) = -\dfrac{1}{4}x\Psi_0[(\Psi_0 - 1)(x - 2x^2) \\ \qquad - 2(1 + x^3 - \Psi_0 x^3)(\ln x + 1 - \ln x)] \\ z_2(x) = -\dfrac{1}{16}x^2\Psi_0^2[2(1 - \Psi_0)x^2 - (1 - \Psi_0)x \\ \qquad - 2(1 + x^3 - 2\Psi_0 x^3)(\ln x + 1 - \ln x)]^2 \\ \vdots \end{cases} \tag{26}$$

The solution of the H-equation Eq. (15) is therefore

$$H(x) = \frac{1}{z(x)} = \frac{1}{\sum_{n=0}^{\infty} z_n} \tag{27}$$

where the $z_n's$ are given in Eq. (26).

The H-functions given by Eq. (27) for various values of Ψ_0 are given in Table 1 using the decomposition method. The values in Table 2 were obtained by Chandrasekhar and Breen (18) by a process of iteration, where the solution started with the fourth approximation for $H(x)$ in terms of the Gaussian division and characteristic roots. The iterates were evaluated at some points and the intermediate values were predicted by interpolating among the differences between the successive iterates. Upon comparing Tables 1 and 2 we note that the decomposition technique with only *three iterations* yields approximately the same values as those in Table 2, derived by Chandrasekhar and Breen, with error less than 1%.

Another satisfactory check for the accuracy of the decomposition method is provided by evaluating the integral

$$\int_0^1 \Psi(t)H(t)dt \tag{28}$$

Table 1. The *H*-Functions Defined in Terms of the Characteristic Function $\Psi(x) = \frac{1}{2}\Psi_0[1 + x(1 - \Psi_0)t^2]$ Are Evaluated Using Adomian's Method with Three Iterations

x	$\Psi_0 = 0.1$	$\Psi_0 = 0.2$	$\Psi_0 = 0.3$	$\Psi_0 = 0.4$	$\Psi_0 = 0.5$
0.00	1.0000	1.0000	1.0000	1.0000	1.0000
0.05	1.0078	1.0158	1.0241	1.0326	1.0415
0.10	1.0125	1.0256	1.0393	1.0538	1.0691
0.15	1.0162	1.0334	1.0517	1.0713	1.0923
0.20	1.0193	1.0400	1.0624	1.0866	1.1129
0.25	1.0220	1.0459	1.0719	1.0004	1.1316
0.30	1.0245	1.0512	1.0806	1.1130	1.1490
0.35	1.0267	1.0561	1.0886	1.1248	1.1653
0.40	1.0288	1.0606	1.0961	1.1359	1.1809
0.45	1.0307	1.0649	1.1032	1.1465	1.1957
0.50	1.0326	1.0690	1.1100	1.1566	1.2100
0.55	1.0343	1.0728	1.1165	1.1663	1.2237
0.60	1.0360	1.0765	1.1227	1.1758	1.2372
0.65	1.0376	1.0801	1.1288	1.1850	1.2503
0.70	1.0391	1.0836	1.1347	1.1939	1.2631
0.75	1.0406	1.0870	1.1404	1.2026	1.2758
0.80	1.0421	1.0902	1.1460	1.2113	1.2882
0.85	1.0435	1.0934	1.1515	1.2198	1.3005
0.90	1.0449	1.0966	1.1570	1.2281	1.3126
0.95	1.0462	1.0997	1.1623	1.2364	1.3248
1.00	1.0475	1.1027	1.1676	1.2446	1.3367

Table 2. The *H*-Functions Defined in Terms of the Characteristic Function $\Psi(x) = \frac{1}{2}\Psi_0[1 + x(1 - \Psi_0)t^2]$ As Obtained by Chandrasekhar and Breen

x	$\Psi_0 = 0.1$	$\Psi_0 = 0.2$	$\Psi_0 = 0.3$	$\Psi_0 = 0.4$	$\Psi_0 = 0.5$
0	1.0000	1.0000	1.0000	1.0000	1.0000
0.05	1.0089	1.0183	1.0280	1.0383	1.0492
0.10	1.0145	1.0297	1.0459	1.0632	1.0817
0.15	1.0188	1.0388	1.0602	1.0832	1.1084
0.20	1.0224	1.0463	1.0722	1.1003	1.1311
0.25	1.0254	1.0528	1.0825	1.1151	1.1511
0.30	1.0280	1.0584	1.0916	1.1281	1.1689
0.35	1.0303	1.0634	1.0996	1.1398	1.1850
0.40	1.0324	1.0678	1.1069	1.1504	1.1996
0.45	1.0343	1.0719	1.1135	1.1600	1.2129
0.50	1.0359	1.0755	1.1194	1.1688	1.2252
0.55	1.0375	1.0788	1.1249	1.1769	1.2365
0.60	1.0389	1.0819	1.1300	1.1844	1.2470
0.65	1.0401	1.0847	1.1346	1.1913	1.2568
0.70	1.0413	1.0873	1.1389	1.1978	1.2659
0.75	1.0424	1.0897	1.1429	1.2038	1.2745
0.80	1.0434	1.0919	1.1467	1.2094	1.2825
0.85	1.0444	1.0940	1.1502	1.2047	1.2900
0.90	1.0453	1.0960	1.1535	1.2196	1.2972
0.95	1.0461	1.0978	1.1566	1.2243	1.3039
1.00	1.0469	1.0995	1.1595	1.2287	1.3103

numerically, where $H(t)$ is the solution obtained from the decomposition technique and then comparing it with its exact value (18) which is given by

$$\int_0^1 \Psi(t)H(t)dt = 1 - \left[1 - 2\int_0^1 \Psi(t)dt\right]^{1/2} \tag{29}$$

For the particular $\Psi(t)$ given in Eq. (16) and upon substituting it into Eq. (29), we get the exact value of the integral

$$\int_0^1 \Psi(t)H(t)dt = 1 - \left[1 - \frac{1}{3}\Psi_0(1 + x - \Psi_0)\right]^{1/2} \tag{30}$$

Table 3 shows that for $x = 1$ and different values of Ψ_0, the error between the exact value in Eq. (30) and the numerical values of the integral in Eq. (28) is less than 1% with H-function being approximated using only three terms of the decomposition method.

KORTEWEG–DE VRIES

In this section a modified decomposition algorithm is presented for solving the Korteweg–de Vries (KdV) equation that

Table 3. Comparison of the Integrals $\int_0^1 \Psi(x)H(x)\,dx$, Where $H(x)$ Is the *H*-Function Obtained Using Decomposition Method with Three Iterations, with Their Exact Values $1 - [1 - \frac{1}{3}\Psi_0(2 - \Psi_0)^{1/2}]$

Ψ_0	Decomposition	Exact	Ψ_0	Decomposition	Exact
0.10	0.06469	0.06726	0.35	0.23252	0.24226
0.15	0.09742	0.10139	0.40	0.26767	0.27889
0.20	0.13051	0.13590	0.45	0.30351	0.31626
0.25	0.16401	0.17084	0.50	0.34040	0.35450
0.30	0.19798	0.20627	0.55	0.37802	0.39378

arises in the study of nonlinear waves. Versions of this equation have been extensively studied both analytically and numerically (21–26). The general KdV equation is given by

$$u_t + f(u)u_x + \beta u_{xxx} = 0 \tag{31}$$

The following choice of the KdV equation which arises in shallow water theory is considered:

$$u_t + (\alpha + \epsilon u)u_x + \beta u_{xxx} = 0 \tag{32}$$

with initial condition

$$u(x, 0) = g(x) \tag{33}$$

where α, β, and ϵ are constants.

Define the linear operators

$$L_t = \frac{\partial}{\partial t}, \quad L_1 = \frac{\partial}{\partial x}, \quad L_2 = \frac{\partial^3}{\partial x^3} \tag{34}$$

The inverse operators are the indefinite integrals given by

$$L_t^{-1} = \int_0^t dt, \quad L_1^{-1} = \int_0^x dx, \quad L_2^{-1} = \int_0^x dx \int_0^x dx \int_0^x dx \tag{35}$$

The conditions under which the decomposition algorithm converges imply the existence of L_t^{-1}, L_1^{-1} and L_2^{-1}. Equation (32) can be written in the following operator form:

$$L_t u = -f(u)L_1 u - \beta L_2 \boldsymbol{u} \tag{36}$$

where

$$f(u) = \alpha + \epsilon u \tag{37}$$

Applying the inverse operator L_t^{-1}, to both sides of Eq. (36) gives

$$L_t^{-1}L_t u = -L_t^{-1}(f(u)L_1 u) - \beta L_t^{-1}L_2 u \tag{38}$$

or

$$u(x, t) = u(x, 0) - L_t^{-1}(f(u)L_1 u) - \beta L_t^{-1}L_2 u \tag{39}$$

Following the decomposition method, the term u_0 is determined as

$$u_0(x, t) = u(x, 0) \tag{40}$$

The other iterations are obtained via:

$$u_{n+1} = -L_t^{-1}[f(u_n)L_1 u_n] - \beta L_t^{-1}L_2 u_n, \quad n \geq 0 \tag{41}$$

Consider now the argument $f(u)L_1u$ of the first term on the right-hand side of Eq. (38). Since

$$u = \sum_{n=0}^{\infty} u_n$$

we have

$$L_1 u = L_1 u_0 + L_1 u_1 + L_1 u_2 + \ldots + L_1 u_n + \ldots \tag{42}$$

Expressing $N(u) = f(u)$ in terms of Adomian polynomials

$$f(u) = A_0 + A_1 + A_2 + \ldots + A_n + \ldots \tag{43}$$

then the product $f(u)L_1u$ can be expanded, after rearranging terms, as follows:

$$\begin{aligned} f(u)L_1 u = {} & [A_0 L_1 u_0] + [A_0 L_1 u_1 + A_1 L_1 u_0] \\ & + [A_0 L_1 u_2 + A_2 L_1 u_0 + A_1 L_1 u_1] \\ & + [A_3 L_1 u_0 + A_2 L_1 u_1 + A_1 L_1 u_2 + A_0 L_1 u_3] \\ & + \ldots \end{aligned} \tag{44}$$

From Eq. (44), the first three *modified Adomian polynomials* $B_i's$ for the nonlinear operator $f(u)u_x = f(u)L_1u = \sum_{n=0}^{\infty} B_n$ are:

$$\begin{cases} B_0 = A_0 L_1 u_0 \\ B_1 = A_0 L_1 u_1 + A_1 L_1 u_0 \\ B_2 = A_0 L_1 u_2 + A_2 L_1 u_0 + A_1 L_1 u_1 \end{cases} \tag{45}$$

Substituting Eq. (44) into Eq. (41) and then replacing the A_n by their values as given in Eq. (10), we obtain the following first three iterates of the *modified Adomian algorithm*

$$\begin{cases} u_1 = -L_t^{-1}[f(u_0)L_1 u_0] - \beta L_t^{-1}L_2 u_0 \\ u_2 = -L_t^{-1}[f(u_0)L_1 u_1 + u_1 f'(u_0)L_1 u_0] - \beta L_t^{-1}L_2 u_1 \\ u_3 = -L_t^{-1}\left[f(u_0)L_1 u_2 + u_1 f'(u_0)L_1 u_1 \right. \\ \qquad \left. + \left(u_2 f'(u_0) + \dfrac{u_1^2}{2!} f''(u_0)\right) L_1 u_0\right] - \beta L_t^{-1}L_2 u_2 \\ \vdots \end{cases} \tag{46}$$

Implementing the modified algorithm to Eqs. (32) and (33), for the special case where $f(u)$ is linear in u and is given in Eq. (37), one obtains

$$u(x, t) = g(x) - L_t^{-1}[(\alpha + \epsilon u)L_1 u] - \beta L_t^{-1}L_2 u \tag{47}$$

It then follows, by using Eqs. (46) and (47), that the iterates of the KdV equation are given by:

$$\begin{cases} u_0 = g(x) \\ u_1 = -L_t^{-1}[(\alpha + \epsilon u_0)L_1 u_0] - \beta L_t^{-1}L_2 u_0 \\ u_2 = -L_t^{-1}[(\alpha + \epsilon u_0)L_1 u_1 + \epsilon u_1 L_1 u_0] - \beta L_t^{-1}L_2 u_1 \\ u_3 = -L_t^{-1}[(\alpha + \epsilon u_0)L_1 u_2 + \epsilon u_1 L_1 u_1 \\ \qquad + (\epsilon u_2)L_1 u_0] - \beta L_t^{-1}L_2 u_2 \\ \vdots \end{cases} \tag{48}$$

We now pick a special function

$$g(t) = A\,\mathrm{sech}^2\left(\sqrt{\frac{\epsilon A}{12\beta}}\,t\right)$$

to compare the exact solution of Eqs. (32) and (33) with the numerical solution obtained by the decomposition method. From Eq. (48) the first two iterates are given by

$$u_0 = A\,\mathrm{sech}^2\left(\sqrt{\frac{\epsilon A}{12\beta}}\,t\right) \tag{49}$$

and

$$u_1 = \frac{1}{9}\frac{1}{\cosh^3\left(\frac{1}{2}\sqrt{\frac{\epsilon A}{3\beta}}t\right)}A(3\alpha+\epsilon A)\sqrt{\frac{3\epsilon A}{\beta}}t\sinh\left(\frac{1}{2}\sqrt{\frac{\epsilon A}{3\beta}}t\right) \tag{50}$$

Many other iterates were generated using MAPLE. Table 4 shows the errors obtained upon solving the KdV equation after normalizing the constants (i.e., we set $\alpha = \beta = A = 1$ and $\epsilon = 0.02$) and using only five iterations of the decomposition method. It is to be noted that only five iterates were needed to obtain an error of less than 10^{-5}%. The overall errors can be made even much smaller by adding new terms of the decomposition.

CONSERVATIVE HYPERBOLIC SYSTEMS

In this section, we will consider the nonlinear partial differential equation:

$$u_t + \frac{\partial}{\partial x}f(u) = 0 \tag{51}$$

$$u(x,0) = h(x) \tag{52}$$

which arises in the formulation of conservative hyperbolic systems. A *hyperbolic system* is one for which the wave speeds coalesce along certain curves in the state space. Such systems occur, for example, in the modeling of oil recovery problems. Many studies have dealt with the numerical diffusion, resolution and shock fronts and spurious oscillations which arise in approximating the solution of Eqs. (51) and (52), (27–29).

Following the decomposition analysis, define the linear operators

$$L_t = \frac{\partial}{\partial t} \tag{53}$$

and

$$L_1 = \frac{\partial}{\partial x} \tag{54}$$

Consequently, Eq. (51) can be written in terms of these operators as

$$L_t u + f'(u)L_1 u = 0 \tag{55}$$

Applying the inverse operator of L_t, namely L_t^{-1}, defined by

$$[L_t^{-1}g](t) := \int_0^t g(y)dy \tag{56}$$

Table 4. Error Obtained Using Decomposition Method with Five Iterations for the KdV Equation

x	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$
0.2	3.1×10^{-9}	4.85×10^{-8}	2.45×10^{-7}	7.79×10^{-7}
0.4	2.8×10^{-9}	4.85×10^{-8}	2.46×10^{-7}	7.78×10^{-7}
0.6	2.9×10^{-9}	4.89×10^{-8}	2.46×10^{-7}	7.76×10^{-7}
0.8	3.1×10^{-9}	4.86×10^{-8}	2.45×10^{-7}	7.74×10^{-7}

to both sides of Eq. (55) yields

$$u(x,t) = u(x,0) - L_t^{-1}[f'(u)L_1u] \tag{57}$$

from which it follows, upon using the initial condition Eq. (52),

$$u(x,t) = h(x) - L_t^{-1}[f'(u)L_1u] \tag{58}$$

If we set

$$N(x) = f(u) = \sum_{n=0}^{\infty}A_n \quad \text{and} \quad u = \sum_{n=0}^{\infty}u_n$$

then the term $f'(u)L_1u$ in Eq. (58) can be expanded in terms of the modified Adomian polynomials B_n's, where

$$f'(u)L_1u = \sum_{n=0}^{\infty}B_n$$

Hence, we have

$$u(x,t) = h(x) - L_t^{-1}\left[\sum_{n=0}^{\infty}B_n\right] \tag{59}$$

To derive the first few B_n's we have

$$\sum_{n=0}^{\infty}B_n = f'(u)L_1u = \left(\sum_{n=0}^{\infty}A'_n\right)\left(\sum_{n=0}^{\infty}L_1u_n\right) \tag{60}$$

and the A'_n are the Adomian polynomials of $f'(u)$. Upon collecting terms in Eq. (60), the first few B_n's are:

$$\begin{aligned} B_0 &= A'_0L_1u_0 \\ B_1 &= A'_0L_1u_1 + A'_1L_1u_0 \\ B_1 &= A'_0L_1u_2 + A'_1L_1u_1 + A'_2L_1u_0 \end{aligned} \tag{61}$$

Applying the decomposition algorithm to Eq. (59), the iterates are given by

$$u_0 = h(x) \tag{62}$$

and

$$u_{n+1} = -L_t^{-1}[B_n], \quad n \geq 1 \tag{63}$$

where the B_n's are given in Eq. (61). Consider the following two special cases:

Case 1. $f(u) = -u^2$

$$u_t - \frac{\partial}{\partial x}u^2 = 0 \tag{64}$$

with initial condition

$$u(x,0) = x \tag{65}$$

Since, for this case,

$$f'(u) = -2u = \sum_{n=0}^{\infty}A'_n$$

hence using Eq. (10) the Adomian polynomials A'_n of $f'(u)$ are given by

$$A'_n = -2u_n, \quad n \geq 0 \tag{66}$$

Using Eqs. (61)–(63) and Eq. (66) the first few iterates are:

$$u_0 = x \tag{67}$$

$$u_1 = 2L_t^{-1}[u_0 L_1 u_0] = 2L_t^{-1}[(x)(1)] = 2xt \tag{68}$$

$$\begin{aligned} u_2 &= 2L_t^{-1}[u_0 L_1 u_1 + u_1 L_1 u_0] \\ &= 2L_t^{-1}[(x)(2t) + (2xt)(1)] = 2L_t^{-1}[4xt] = 4xt^2 \end{aligned} \tag{69}$$

Similarly,

$$\begin{aligned} u_3 &= 2L_t^{-1}[u_0 L_1 u_2 + u_1 L_1 u_1 + u_2 L_1 u_0] \\ &= 2L_t^{-1}[(x)(4t^2) + (2xt)(2t) + (4xt^2)(1)] = 24L_t^{-1}[xt^2] = 8xt^3 \end{aligned} \tag{70}$$

Summing these iterates yields

$$u = x + 2xt + 4xt^2 + 8xt^3 + 16xt^4 + \ldots \tag{71}$$

If $-\frac{1}{2} < t < \frac{1}{2}$, then Eq. (71) can be written in closed form as

$$u(x,t) = \frac{x}{1-2t} \tag{72}$$

which is the exact solution of Eqs. (64) and (65).

Case 2. $f(u) = \cos u$

$$u_t + \frac{\partial}{\partial x}\cos u = 0 \tag{73}$$

with initial condition

$$u(x,0) = x \tag{74}$$

For this case

$$f'(u) = -\sin u = \sum_{n=0}^{\infty} A'_n$$

hence using Eq. (10) the Adomian polynomials A'_n of $f'(u)$ are given by

$$\begin{aligned} A'_0 &= -\sin u_0 \\ A'_1 &= -u_1 \cos u_0 \\ A'_2 &= -u_2 \cos u_0 + \frac{1}{2!} u_1^2 \sin u_0 \end{aligned} \tag{75}$$

Using Eq. (61), Eq. (63), and (75) gives the following first few iterates

$$u_0 = x \tag{76}$$

$$u_1 = L_t^{-1}[\sin u_0 L_1 u_0] = L_t^{-1}[(\sin x)(1)] = t \sin x \tag{77}$$

$$\begin{aligned} u_2 &= L_t^{-1}[u_1 \cos u_0 L_1 u_0 + \sin u_0 L_1 u_1] \\ &= L_t^{-1}[(t \sin x)(\cos x)(1) + (\sin x)(t \cos x)] \\ &= L_t^{-1}[2t \sin x \cos x] = \frac{1}{2} t^2 \sin 2x \end{aligned} \tag{78}$$

Similarly,

$$\begin{aligned} u_3 &= L_t^{-1}\Big[\sin u_0 L_1 u_2 + u_1 \cos u_0 L_1 u_1 \\ &\quad + \left(u_2 \cos u_0 - \frac{1}{2} u_1^2 \sin u_0\right) L_1 u_0\Big] \\ &= \frac{1}{3} t^3 \left(\sin 3x - \frac{1}{2}\sin^3 x\right) \end{aligned} \tag{79}$$

Upon summing these iterates we get

$$u = x + \sin xt + \frac{1}{2}\sin 2xt^2 + \frac{1}{3}\left(\sin 3x - \frac{1}{2}\sin^3\right) t^3 + \ldots \tag{80}$$

KLEIN–GORDON EQUATION

In this section, we will focus on the nonlinear Klein–Gordon equation given generally by:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \Delta u(x,t) + ku(x,t) + f(u(x,t)) = g(x,t) \tag{81}$$

$$u(x,0) = b_0(x), \quad \frac{\partial u}{\partial t}(x,0) = b_1(x) \tag{82}$$

with

$$x = (x_1, x_2, \ldots, x_m) \in R^m, \quad t \in (0,T]$$

where

$$\Delta = \sum_{j=0}^{m} \frac{\partial^2}{\partial x_j^2} \tag{83}$$

and k is real, f is a given nonlinear function, and h is a known function.

The Klein–Gordon equation is an important mathematical model in quantum mechanics and also occurs in relativistic physics as a model of dispersive phenomena [see (25,26,28–33)].

Following the decomposition scheme, define

$$L_t = \frac{\partial^2}{\partial t^2}, \quad L_{x_i} = \frac{\partial^2}{\partial x_i^2}, \quad i = 1, 2, \ldots, m \tag{84}$$

Consequently, Eq. (81) can be written in the following operator form

$$L_t u = \sum_{i=1}^{m} L_{x_i} u - ku - f(u) + g \tag{85}$$

It was shown in (24) that Eq. (81) with the conditions of Eq. (82) possesses a unique solution. Thus the inverse operator of L_t, namely L_t^{-1}, exists and is the two-fold indefinite integral, that is,

$$[L_t^{-1}h](t) := \int_0^t du \int_0^u dv h(v) \tag{86}$$

Applying L_t^{-1}, to both sides of Eq. (84) yields

$$L_t^{-1}L_t u = \sum_{i=0}^{m} L_t^{-1}L_{x_i}u - kL_t^{-1}u - L_t^{-1}(f(u)) + L_t^{-1}g \tag{87}$$

and upon using the initial conditions of Eq. (82) it follows that

$$\begin{aligned} u(x,t) = b_0(x) + b_1(x)t + \sum_{i=0}^{m} L_t^{-1}L_{x_i}u - kL_t^{-1}u \\ - L_t^{-1}(f(u)) + L_t^{-1}g \end{aligned} \tag{88}$$

Following the decomposition technique the first term u_0 is determined as

$$u_0(x,t) = b_0(x) + b_1(x)t + L_t^{-1}(g(x,t)) \tag{89}$$

Setting $N(u) = f(u) = \sum_{n=0}^{\infty} A_n$, then the next iterates are determined as

$$u_{n+1} = \sum_{i=0}^{m} L_t^{-1}L_{x_i}u_n - kL_t^{-1}u_n - L_t^{-1}A_n, \quad n \geq 0 \tag{90}$$

Replacing the A_n in Eq. (90) by their values as given in Eq. (10), then the first three iterates are given by

$$\begin{cases} u_1 = \sum_{i=0}^{m} L_t^{-1}L_{x_i}u_0 - kL_t^{-1}u_0 - L_t^{-1}(f(u_0)) \\ u_2 = \sum_{i=0}^{m} L_t^{-1}L_{x_i}u_1 - kL_t^{-1}u_1 - L_t^{-1}(u_1 f(u_0)) \\ u_3 = \sum_{i=0}^{m} L_t^{-1}L_{x_i}u_2 - kL_t^{-1}u_2 \\ \qquad - L_t^{-1}\left(u_2 \dfrac{d}{du_0}f(u_0) + \dfrac{u_1^2}{2!}\dfrac{d^2}{du_0^2}f(u_0)\right) \end{cases} \tag{91}$$

We will show through two examples that the number of terms required to obtain an accurate computable solution is very small. The outcome of the decomposition method will be compared with the known solution to the underlying Klein–Gordon equation. The solutions obtained are generated by using MAPLE.

Example 1. Consider the Klein–Gordon equation of the form

$$\begin{aligned} u_{tt} - u_{xx} + \frac{\pi^2}{4}u + u^3 = \frac{x^3}{4}\cos\frac{3\pi}{2}t + \frac{3x^3}{4}\cos\frac{\pi}{2}t, \\ u(x,0) = x, u_t(x,0) = 0 \end{aligned} \tag{92}$$

Equation (88) implies that

$$\begin{aligned} u(x,t) = x + L_t^{-1}L_x u - \frac{\pi^2}{4}L_t^{-1}u - L_t^{-1}u^3 \\ + L_t^{-1}\left(\frac{x^3}{4}\cos\frac{3\pi}{2}t + \frac{3x^3}{4}\cos\frac{\pi}{2}t\right) \end{aligned} \tag{93}$$

Table 5. Error Obtained Using Decomposition Method with Three Iterations for the Klein–Gordon Eq. (92)

x	$t = 0.1$	$t = 0.3$	$t = 0.5$
0.1	8.1×10^{-9}	5.8×10^{-6}	1.1×10^{-4}
0.3	2.1×10^{-8}	1.5×10^{-5}	3.1×10^{-4}
0.5	2.2×10^{-8}	1.6×10^{-5}	3.4×10^{-4}

Equations (89) and (91) imply that the various iterates are given by:

$$\begin{aligned} u_0 &= x + L_t^{-1}\left(\frac{x^3}{4}\cos\frac{3}{2}\pi t + \frac{3x^3}{4}\cos\frac{\pi}{2}t\right) \\ &= x + \frac{1}{9\pi^2}x^3\left(28 - \cos\frac{3}{2}\pi t + 27\cos\frac{1}{2}\pi t\right) \end{aligned} \tag{94}$$

and

$$u_1 = L_t^{-1}L_x u_0 - \frac{\pi^2}{4}L_t^{-1}u_0 - L_t^{-1}u_0^3 \tag{95}$$

In a like manner,

$$u_2 = L_t^{-1}L_x u_1 - \frac{\pi^2}{4}L_t^{-1}u_1 - 3L_t^{-1}u_0^2 u_1 \tag{96}$$

Table 5 shows the error obtained by comparing the decomposition method with three iterations and the exact solution which is $u = x\cos\pi/2t$.

Example 2. Consider the Klein–Gordon equation of the form

$$\begin{aligned} u_{tt} - u_{xx} + \pi^2 u + u^2 = x^4\cos^2\pi t - 2\cos\pi t, \\ u(x,0) = x^2, u_t(x,0) = 0 \end{aligned} \tag{97}$$

Equation (88) implies that

$$\begin{aligned} u(x,t) = x^2 + L_t^{-1}L_x u - \pi^2 L_t^{-1}u - L_t^{-1}u^2 \\ + L_t^{-1}(x^4\cos^2\pi t - 2\cos\pi t) \end{aligned} \tag{98}$$

Equations (89) and (91) imply that the various iterates are given by:

$$\begin{aligned} u_0 &= x^2 + L_t^{-1}(x^4\cos^2\pi t - \cos\pi t) \\ &= -\frac{2}{\pi^2} + x^2 + \frac{1}{4\pi^2}(\pi^2 x^4 t^2 + 8\cos\pi t + x^4\sin^2\pi t) \end{aligned} \tag{99}$$

and

$$u_1 = L_t^{-1}L_x u_0 - \pi^2 L_t^{-1}u_0 - L_t^{-1}u_0^2 \tag{100}$$

In a like manner,

$$u_2 = L_t^{-1}L_x u_1 - \pi^2 L_t^{-1}u_1 - 2L_t^{-1}u_0 u_1 \tag{101}$$

Table 6. Error Obtained Using Decomposition Method with Three Iterations for the Klein–Gordon Eq. (97)

x	$t = 0.1$	$t = 0.3$	$t = 0.5$
0.1	7.3×10^{-7}	5.2×10^{-4}	1.1×10^{-2}
0.3	7.0×10^{-7}	5.0×10^{-4}	1.0×10^{-2}
0.5	6.0×10^{-7}	4.2×10^{-4}	8.7×10^{-3}

Table 6 shows the error obtained by comparing the decomposition method with three iterations and the exact solution which is $u = x^2 \cos \pi t$.

Many other interesting physical problems whose mathematical formulation lead to nonlinear equations can be handled by the decomposition method. Some open problems related to this decomposition method include an extension of the method to solve nonlinear equations with specified behavior at ∞ and nonlinear equations with fractional exponents.

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NONLINEAR EQUATIONS. See Nonlinear equations.
NONLINEAR FILTERING. See Filtering and estimation, nonlinear.