SWITCHING FUNCTIONS

In this article, we begin by illustrating the concept of universal adaptive control by considering a simple class of scalar systems and also motivate the use of switching functions for this class. We then present Nussbaum functions. These arise naturally in the feedback law if the sign of the high-frequency gain of the system to be stabilized is unknown. An alternative to Nussbaum functions are switching decision functions which are considered in the next section. Then we discuss switching functions and unbounded switching functions, respectively. Finally, we give a brief overview of how the switching functions described above are related and used to solve the universal adaptive control problem for different classes of systems.

UNIVERSAL ADAPTIVE CONTROL

Simplified and loosely speaking, in universal adaptive control we consider a *class* of systems of the form

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad y(t) = h(t, x(t)) \tag{1}$$

satisfying certain structural assumptions, and we want to design a single feedback law

$$u(t) = K_{k(t)} y(t) \tag{2}$$

and an adaptation law

$$k(t) = \varphi(t, y(\cdot)) \tag{3}$$

so that if Eqs. (2) and (3) are applied to Eq. (1), then the closed-loop system has bounded signals and meets certain other control objectives; for example, $\lim_{t\to\infty} y(t) = 0$. No identification mechanisms or probing signals should be incorporated.

If we restrict our attention to universal adaptive controllers that do not use any observers, then this approach was introduced for linear minimum phase systems by the seminal work of Byrnes (1), Mareels (2), Morse (3), and Willems (4) in the early 1980s.

To understand the idea, consider, instead of Eq. (1), the class of scalar systems

$$\dot{x}(t) = ax(t) + bu(t), \qquad y(t) = cx(t), \qquad x(0) = x_0 \tag{4}$$

where a, b, c, $x_0 \in \mathbb{R}$ are unknown and the only structural knowledge is $cb \neq 0$. Suppose, for a moment, the stronger assumption $cb > 0$, that is, the sign of the high-frequency gain is known, and apply

$$u(t) = -k(t)y(t) \tag{5}$$

$$\dot{k}(t) = y(t)^2 \tag{6}$$

Note that Eqs. (5) and (6) are a very simple specification of Eqs. (2) and (3), and they consist of a time-varying proportional output feedback and a monotonically nondecreasing gain adaptation. The closed-loop system becomes

$$\dot{x}(t) = [a - k(t)cb]x(t) \tag{7}$$

$$\dot{k}(t) = c^2x(t)^2 \tag{8}$$

As long as Eq. (7) is not exponentially stable, $|x(t)|$ will grow and therefore $k(t)$ will grow. Finally, $k(t)$ becomes so large that Eq. (7) is exponentially stable, and then exponential decay of $|x(t)|$ also ensures that $k(t)$ converges to a finite limit as t tends to ∞.

Morse (3) raised the question whether the knowledge of the sign of the high-frequency gain of single-input, single-output, minimum phase systems is a necessary information to achieve stabilization. For the above example, this means whether one can achieve stabilization if $cb \neq 0$. If $cb < 0$, then obviously Eq. (5) fails because the system [Eq. (7)] becomes unstable. So if the sign of cb is unknown, one has to search adaptively for the correct sign. This was achieved by Nussbaum's contribution (5), which suggested that we modify the feedback law [Eq. (5)] as follows:

$$u(t) = -k(t)\cos\sqrt{k(t)}y(t) \tag{9}$$

In fact, Nussbaum (5) presented a more general but more complicated solution. However, Eq. (9) captures the essence and is easier to understand. The intuition behind the fact that Eqs. (6) and (9) comprise a universal adaptive controller of the class Eq. (4) with $cb \neq 0$ follows: The controller has to find by itself the correct sign so that the feedback equation [Eq. (9)] stabilizes Eq. (4). The function $\cos\sqrt{k(t)}$ in Eq. (9) is responsible for the search of the sign; and while $k(t)$ in Eq. (9) is monotonically increasing, it switches sign. If the sign is "correct" (i.e., $\operatorname{sgn}\cos\sqrt{k(t)} = \operatorname{sgn} cb$) and the gain is sufficiently large, then $\dot{x}(t) = [a - cb\,k(t)\cos\sqrt{k(t)}]x(t)$ is exponentially stable and $|x(t)|$ decays to zero exponentially. If the convergence is sufficiently fast so that $k(t) = k(0) + \int_0^t y(\tau)^2\,d\tau$ converges without becoming so large that $\cos\sqrt{k(t)}$ changes sign again, then the closed-loop system remains stable. The latter is ensured by the square root in $\cos\sqrt{k}$.

To see this and also to gain a deeper understanding of the general nature of this switching function approach, we sketch the proof of the universal adaptive stabilization. Observe that the closed-loop system consisting of Eqs. (4), (6), and (9) satisfies

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}y(t)^2 &= y(t)\dot{y}(t) = [a - cbk(t)\cos\sqrt{k(t)}]y(t)^2 \\ &= [a - cbk(t)\cos\sqrt{k(t)}]\dot{k}(t)\end{aligned}$$

and integration together with the substitution $k(\tau) = \mu$ yields, provided that $k(t) > k(0)$:

$$\begin{aligned}\frac{1}{2}y(t)^2 - \frac{1}{2}y(0)^2 &= \int_0^t [a - cbk(\tau)\cos\sqrt{k(\tau)}]\dot{k}(\tau)\,d\tau \\ &= \int_{k(0)}^{k(t)} [a - cb\,\mu\,\cos\sqrt{\mu}]\,d\mu \\ &= [k(t) - k(0)] \\ &\quad \times \left[a - \frac{cb}{k(t)-k(0)}\int_{k(0)}^{k(t)} \mu\cos\sqrt{\mu}\,d\mu\right]\end{aligned} \tag{10}$$

Seeking a contradiction, suppose that $k(t)$ tends to ∞ as t goes to ∞ [note that by Eq. (6), $t \mapsto k(t)$ is monotonically nondecreasing]. Since

$$\frac{1}{k}\int_0^k \mu\cos\sqrt{\mu}\,d\mu = \frac{2}{k}\int_0^k \tau^3\cos\tau\,d\tau \tag{11}$$

takes arbitrary large positive and negative values as $k \to \infty$, we derive a contradiction at Eq. (10). Therefore $k(\cdot)$ must be bounded. This is equivalent to $y \in L_2(0, \infty)$. Using Eq. (7) gives $\dot{y} \in L_2(0, \infty)$. Now by a simple argument it follows that $\lim_{t\to\infty} y(t) = 0$.

The property that the function in Eq. (11) takes arbitrarily large positive and negative values as $k \to \infty$ is crucial and will be considered more generally in the following section.

NUSSBAUM FUNCTIONS

If the underlying class of systems consists of linear, multi-input, multi-output systems

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) \tag{12}$$

where $A \in \mathbb{R}^{n\times n}$, $B, C^T \in \mathbb{R}^{n\times m}$ and the structural assumptions are minimum phase and

$$\sigma(CB) \subset \mathbb{C}_+ \quad \text{or} \quad \sigma(CB) \subset \mathbb{C}_- \tag{13}$$

then it is well known that static output feedback

$$u(t) = -Sky(t)$$

stabilizes Eq. (12) provided that k is sufficiently large and the sign is correct; that is, $S = +1$ if $\sigma(CB) \subset \mathbb{C}_+$ and $S = -1$ otherwise. If the sign is unknown, and that is what we assume in Eq. (13), then it has to be found adaptively similarly as described in the section entitled "Universal Adaptive Control." The feedback law [Eq. (2)] now becomes

$$u(t) = -N(k(t))y(t) \tag{14}$$

where $N(\cdot)$ captures the essential features of the function $k \mapsto k\cos\sqrt{k}$, and the gain adaptation [Eq. (3)] becomes

$$\dot{k}(t) = \|y(t)\|^2 \tag{15}$$

Now Eqs. (14) and (15) comprise a universal adaptive stabilizer for the class consisting of Eqs. (12) and (13) of minimum-phase systems if $N(\cdot)$ is a Nussbaum function defined as follows; see Nussbaum (5).

Definition 1. A piecewise right continuous and locally Lipschitz function $N(\cdot):[0, \infty) \to \mathbb{R}$ is called a *Nussbaum function* if, and only if, it satisfies

$$\limsup_{k>0} \frac{1}{k}\int_0^k N(\tau)\,d\tau = +\infty \quad \text{and} \quad \liminf_{k>0} \frac{1}{k}\int_0^k N(\tau)\,d\tau = -\infty \tag{16}$$

It is easy to see that Eq. (16) implies that, for every $k_0 \in (0, \infty)$,

$$\limsup_{k>k_0} \frac{1}{k-k_0}\int_{k_0}^k N(\tau)\,d\tau = +\infty$$

and

$$\liminf_{k>k_0} \frac{1}{k-k_0}\int_{k_0}^k N(\tau)\,d\tau = -\infty$$

Example 1. The following functions are Nussbaum functions:

$$\begin{aligned}
&N_1(k) = k^2\cos k, && k \in \mathbb{R}\\
&N_2(k) = k\cos\sqrt{|k|}, && k \in \mathbb{R}\\
&N_3(k) = \ln k \cos\sqrt{\ln k}, && k > 1\\
&N_4(k) = \begin{cases} k & \text{if } n^2 \le |k| < (n+1)^2, \quad n \text{ even},\\ -k & \text{if } n^2 \le |k| < (n+1)^2, \quad n \text{ odd}, \end{cases} && k \in \mathbb{R}\\
&N_5(k) = \begin{cases} k & \text{if } \tau_n \le |k| < \tau_0\\ k & \text{if } \tau_n \le |k| < \tau_{n+1}, \quad n \text{ even}\\ -k & \text{if } \tau_n \le |k| < \tau_{n+1}, \quad n \text{ odd} \end{cases}\\
&\qquad \text{with } \tau_0 > 1,\ \tau_{n+1} := \tau_n^2, && k \in \mathbb{R}\\
&N_6(k) = \cos\left(\frac{\pi}{2}k\right)\cdot e^{(k^2)}, && k \in \mathbb{R}
\end{aligned}$$

Of course, the cosine in the above examples can be replaced by the sine. It is easy to see that $N_1(k)$, $N_2(k)$, $N_4(k)$, and $N_5(k)$ are Nussbaum functions. For a proof for $N_3(k)$ and $N_6(k)$ see Refs. 6 and 7, respectively.

$N_3(k)$ was successful if Eq. (12) consists of single-input, single-output, high-gain stabilizable systems of relative degree two (Ref. 6), and is also important when the output is sampled (Ref. 8). The function has the property that the intervals where the sign is kept constant are increasing. In fact we have $\lim_{k\to\infty} (d/dk)N_3(k) = 0$.

If the system class is subjected to actuator and sensor nonlinearities, then Eq. (16) is too weak. Therefore Logemann and Owens (7) introduced the following more restrictive concept.

Definition 2. A Nussbaum function $N(\cdot):[0, \infty) \to \mathbb{R}$ is called *scaling-invariant* if, and only if, for arbitrary $\alpha, \beta > 0$, we have

$$\tilde{N}(t) := \begin{cases} \alpha N(t) & \text{if } N(t) \ge 0\\ \beta N(t) & \text{if } N(t) < 0 \end{cases}$$

is a Nussbaum function, too.

Scaling invariance of $N_6(k)$ is proved in Ref. 7.

SWITCHING DECISION FUNCTIONS

An alternative approach to the Nussbaum switching strategy is via a switching decision function as introduced by Ilchmann and Owens (9). As in the section entitled "Nussbaum Functions," consider the class of minimum phase systems [Eq. (12)] satisfying Eq. (13). The gain adaptation [Eq. (15)] can be slightly generalized by

$$\dot{k}(t) = \|y(t)\|^p \tag{17}$$

where $p \ge 1$, and Eq. (14) is replaced by

$$u(t) = -k(t)\Theta(t)y(t) \tag{18}$$

where $\Theta(\cdot)$ is defined as follows: Let $0 < \lambda_1 < \lambda_2 < \cdots$ be a strictly increasing sequence with $\lim_{i\to\infty} \lambda_i = \infty$ and define the function

$$\Theta(\cdot) : [0, \infty) \to \{-1, +1\}$$

by the *switching decision function*

$$\psi(t) = \frac{k_0 + \int_0^t \Theta(\tau)k(\tau)\|y(\tau)\|^p\,d\tau}{1 + \int_0^t \|y(\tau)\|^p\,d\tau}$$

and the algorithm

$$\begin{aligned}
&i := 0\\
&\Theta(0) := -1, \qquad t_0 := 0\\
(*)\quad &t_{i+1} := \inf\{t > t_i \,|\, |\psi(t)| \le \lambda_{i+1}k(0)\}\\
&\Theta(t) := \Theta(t_i) \qquad \text{for all} \quad t \in [t_i, t_{i+1})\\
&\Theta(t_{i+1}) := -\Theta(t_i)\\
&i := i+1\\
&\text{go to } (*)
\end{aligned} \tag{19}$$

Then equations (17)–(19) comprise a universal adaptive stabilizer for the class of minimum phase systems [Eq. (12)] which satisfy Eq. (13). The intuition behind this control relies on the fact that the switching function $\Theta(\cdot)$ switches at each time t_i when the switching decision function $\psi(\cdot)$, which is a stability indicator, reaches the 'threshold' $\lambda_{i+1}k(0)$.

For $k(t) \ge k(0) > 0$, it is easy to see that, for every $t \ne t_i$, we obtain

$$\frac{d}{dt}\psi(t) = \begin{cases} \ge 0 & \text{if } \Theta(t) = +1\\ \le 0 & \text{if } \Theta(t) = -1 \end{cases}$$

It can be shown that if $k(t)$ is strictly increasing, then $\psi(t)$ is either strictly increasing or decreasing, taking larger negative and positive values. Therefore, by Eq. (17), the gain $k(t)$ will increase and, by Eq. (19), $\Theta(\cdot)$ will keep on switching, until finally $k(t)$ will be so large and the sign of $\Theta(t)$ will be correct, so that the system will be stabilized and $\Theta(t)$ will not switch sign again.

The advantage of this strategy, when compared to the Nussbaum-type switching strategy, is that the "stability indicator" $\psi(t)$ is more strongly related to the dynamics of the system and the controller tolerates large classes of nonlinear disturbances. Note also that no assumption is made on how fast the sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ is tending to ∞.

The close relationship between the concept of switching decision functions and Nussbaum functions is made precise in the following lemma; a proof is given in Refs. 9 and 10.

Lemma 1. Consider Eq. (12) and suppose $\dot{k}(t) = \|y(t)\|^p > 0$ almost everywhere and $k(\cdot)$ is unbounded. Then the inverse functions $\tau \mapsto k^{-1}(\tau)$ is well-defined on $[0, \infty)$, $\psi(t)$ takes arbitrary large negative and positive values, and $\eta \mapsto (\Theta \circ k^{-1})(\eta) \cdot \eta$ is a Nussbaum function.

SWITCHING FUNCTIONS

For the more general class of systems [Eq. (12)] where, instead of Eq. (13), it is only assumed that

$$\det(CB) \neq 0$$

Mårtensson (11) introduced

$$u(t) = -k(t)K_{(S\circ k)(t)}y(t) \tag{20}$$

to replace Eq. (14). Suppose $K_{(S\circ k)(\cdot)} \equiv K \in \mathbb{R}^{m\times m}$ so that $\sigma(CBK) \subset \mathbb{C}_+$, then Eq. (20) obviously stabilizes each system (12) provided that $k(\cdot) \equiv k \in \mathbb{R}$ is sufficiently large.

Such a K belongs to the so-called *finite spectrum unmixing set*—that is, a set

$$\{K_1, \ldots, K_N\} \subset GL_m(\mathbb{R})$$

so that, for any $M \in GL_m(\mathbb{R})$ there exists $i \in \{1, \ldots, N\}$ such that

$$\sigma(MK_i) \subset \mathbb{C}_+$$

The existence of this set was proved in Ref. 12. Now in the adaptive setup K is unknown and therefore $K_{(S\circ k)(t)}$ has to travel through the finite spectrum unmixing set and stay sufficiently long with the system to give it enough time to settle down. This is a similar scenario as in the single-input, single-output case ($m = 1$) where the set $\{1, -1\}$ is obviously unmixing.

In general the switching is achieved by the following function.

Definition 3. Let $N \in \mathbb{N}$. If the sequence $0 < \tau_1 < \tau_2 < \ldots$ satisfies $\lim_{i\to\infty} \tau_i = \infty$, then the associated function

$$S(\cdot): \mathbb{R} \to \{1, \ldots, N\}, k \mapsto S(k) = \begin{cases} 1 & \text{if } k \in (-\infty, \tau_1) \\ i, & \text{if } k \in [\tau_{lN+i}, \tau_{lN+i+1}) \\ & \quad \text{for some } l \in \mathbb{N}_0, i \in \{1, \ldots, N\} \end{cases}$$

is called a *switching function.*

As for Nussbaum functions, the growth of the switching points τ_i is important, and quite often a growth condition such as

$$\lim_{i\to\infty} \frac{\tau_{i-1}}{\tau_i} = 0 \tag{21}$$

is needed.

Obviously, if $\{\tau_i\}_{i\in\mathbb{N}}$ satisfies Eq. (21), then $\lim_{i\to\infty} \tau_i = \infty$. An example for a sequence satisfying Eq. (21) is $\tau_{i+1} := \tau_i + e^{(i^2)}$; see Ref. 13.

However, the cardinality of the unmixing set can be very large. For $m = 2$ there exists an unmixing set of cardinality 6, and $GL_3(\mathbb{R})$ can be unmixed by a set with cardinality 32; see Ref. 14. Hardly anything is known on the minimum cardinality of unmixing sets for $m > 3$; see Ref. 12.

The relationship between a Nussbaum function and a switching function is given in the following lemma; for a proof see Refs. 10 and 13.

Lemma 2

1. If $S(\cdot): \mathbb{R} \to \{1, 2\}$ is a switching function with associated sequence $\{\tau_i\}_{i\in\mathbb{N}}$ satisfying Eq. (21), then

$$N(k) = k \cdot K_{S\circ k}$$

   is a Nussbaum function, where $K_1 := 1$, $K_2 := -1$, that is, a spectrum unmixing set for $\mathbb{R}\backslash\{0\}$.

2. Suppose $S(\cdot): \mathbb{R} \to \{1, \ldots, N\}$, $N \in \mathbb{N}$, is a switching function associated with $\{\tau_i\}_{i\in\mathbb{N}}$ satisfying Eq. (21). Then, for arbitrary $\alpha > 0$ and every $i \in \{1, \ldots, N\}$, the function

$$F_i^\alpha(\cdot): \mathbb{R} \to \mathbb{R}, \qquad k \mapsto \begin{cases} k & \text{if } \ S(k) = i \\ -\alpha k & \text{if } \ S(k) \neq i \end{cases}$$

   is a scaling-invariant Nussbaum function.

UNBOUNDED SWITCHING FUNCTIONS

If even the minimum phase assumption for systems of the form presented in Eq. (12) is dropped and the only structural assumption being made is that for each system there exists a stabilizing output feedback $u(t) = -Ky(t)$ for some $K \in \mathbb{R}^{m\times m}$, then Mårtensson (11) introduced the feedback

$$u(t) = -k(t)K_{(\sigma\circ k)(t)}y(t) \tag{22}$$

Now $t \mapsto K_{(\sigma \circ k)(t)}$ has to travel through a countable set of controllers $\{K_i\}_{i\in\mathbb{N}}$ which contains some $K \in R^{m\times m}$ so that $u(t) = -Ky(t)$ stabilizes Eq. (12). $\{K_i\}_{i\in\mathbb{N}}$ could be, for example, $\mathbb{Q}^{m\times m}$.

The problem is again that $K_{(\sigma \circ k)(t)}$ stays sufficiently long at K so that the output converges to zero sufficiently fast to ensure that no more switchings occur. Otherwise, $(\sigma \circ k)(t)$ has to ensure that $K_{(\sigma \circ k)(t)}$ comes back to a neighborhood of K and this time stays even longer there. The property of "coming back" is achieved by requiring $\sigma(\cdot)$ to be an unbounded switching function defined as follows.

Definition 4. Suppose $0 < \tau_1 < \tau_2 < \ldots$ is a sequence satisfying $\lim_{i\to\infty} \tau_i = \infty$. A right continuous function $\sigma(\cdot): \mathbb{R} \to \mathbb{N}$ is called an *unbounded switching function* with discontinuity points $\{\tau_i\}$ if, and only if, for all $a \in \mathbb{R}$, $\sigma([a, \infty)) = \mathbb{N}$.

In the literature an unbounded switching function is mostly called switching function, but here we like to emphasize the difference between a switching function and an unbounded switching function.

As in the case of switching and Nussbaum functions, the growth of the switching points is important and ensures that the system stays sufficiently long with a possibly stabilizing feedback. If we consider the class of systems described at the beginning of this section, then Eq. (22) together with the gain adaptation

$$\dot{k}(t) = \|y(t)\|^2 + \|u(t)\|^2$$

is a universal adaptive stabilizer provided that $\sigma(\cdot)$ is an unbounded switching function, the discontinuity points are given by $\tau_{i+1} = \tau_i^2$, $\tau_1 > 1$, and $\{K_i\}_{i\in\mathbb{N}} = \mathbb{Q}^{m\times m}$; for a proof see Refs. 11 and 15.

Very closely related to this concept are the so-called *tuning functions* used by Miller and Davison, who extended Mårtensson's approach considerably; for a survey of their work see Ref. 16.

APPLICATIONS

In recent years the concepts discussed above have been pushed much further for applications in adaptive control. A sophisticated switching strategy called *cyclic switching* was introduced by Morse and Pait (17,18) to solve stabilization problems which arise in the synthesis of identifier-based adaptive control. The scope of so-called *logic-based switching controllers* was discussed at a recent workshop, and many different approaches are encompassed in Ref. 19.

In the previous sections we have motivated the use of Nussbaum functions (NFs), switching decision functions (SDFs) switching functions (SFs), and unbounded switching functions (USFs) for different linear system classes. Survey articles on this subject are Refs. 10 and 20 for finite-dimensional systems and Ref. 21 for infinite-dimensional systems. In the following we relate these functions to various other classes that they have been used for and give references to where they have been studied. We only consider continuous-time systems. There are a few results available which make use of switching functions in adaptive control of discrete-time systems.

The acronym SISO is used for single-input, single-output systems, and the acronym MIMO is used for multi-input, multi-output systems.

The following first three lists are only concerned with universal adaptive *stabilization* of *minimum-phase* systems.

Linear, Finite-Dimensional, Minimum-Phase Systems

- NF: SISO, $cb \neq 0$: (4,22)
- NF: SISO, relative degree 2: (6,23,24)
- NF: SISO, $cb \neq 0$, exponential stabilization: (25)
- NF: SISO, $cb \neq 0$, nonlinear perturbations: (26,27)
- NF: MIMO, $\sigma(CB) \subset \mathbb{C}_-$ or $\subset \mathbb{C}_+$, exponential stabilization: (28)
- SDF: SISO, $cb \neq 0$: (29)
- SF: MIMO, $\det(CB) \neq 0$, exponential stabilization: (13)
- SDF: MIMO, $\sigma(CB) \subset \mathbb{C}_-$ or $\subset \mathbb{C}_+$, nonlinear perturbations: (9)

Linear, Infinite-Dimensional, Minimum-Phase Systems

- NF: SISO: (30–33)
- NF: SISO, nonlinear perturbations: (7,34)
- NF: SISO, sector-bounded perturbations, exponential stabilization: (35)
- SF: MIMO, $\det(CB) \neq 0$: (36)

Nonlinear Systems, Stabilization

- NF: scalar: (37)
- NF: SISO, homogeneous: (38)

Discontinuous-Feedback, Finite-Dimensional, Minimum-Phase Systems

- SF: MIMO, linear, stabilization: (39)
- NF: SISO, nonlinear, stabilization: (40–42)
- NF: SISO, λ-tracking, nonlinear perturbations: (42–44)

So far the above articles all deal with stabilization. In the following we also consider asymptotic tracking of reference signals produced by a known linear finite-dimensional differential equation.

Tracking With Internal Model

- NF: MIMO, $\sigma(CB) \subset \mathbb{C}_-$ or $\subset \mathbb{C}_+$, experimental tracking: (28)
- SF: MIMO, $\det(CB) \neq 0$: (36)
- NF: SISO, $cb \neq 0$, relative degree 1 or 2: (45–47)
- NF: SISO, $cb \neq 0$, relative degree known: (48)
- NF: MIMO, $\sigma(CB) \subset \mathbb{C}_-$ or $\subset \mathbb{C}_+$: (49)
- SF: MIMO, $\det(CB) \neq 0$: (49)

In the following we consider λ-tracking of bounded reference signals with bounded derivatives. λ-tracking means that the tracking error converges to a ball around zero of prespecified radius $\lambda > 0$.

λ-Tracking, Continuous-Feedback, Minimum-Phase Systems

NF: SISO, piecewise constant gain: (50)

NF: SISO, linear, continuous gain: (51)

NF: SISO, nonlinear, continuous gain: (42,52)

Topological Aspects

SF: finite-dimensional linear, SISO, minimum phase, stabilization: (53,54)

SF: finite-dimensional linear, MIMO, $\sigma(CB) \neq 0$, minimum phase, tracking: (55)

USF: finite-dimensional linear, MIMO, nonminimum phase: (56)

SF and NF: scalar linear, exact solutions: (22,57)

Non-Minimum-Phase Systems, Stabilization

SF: MIMO, linear, stabilization: (11,58–60)

USF: MIMO, constant reference signals: (61)

USF: MIMO, linear, stabilization: (61)

USF: MIMO, tracking with internal model: (62)

USF: stable MIMO, low gain, tracking constant signals: (63)

SF & NF: stable infinite-dimensional MIMO, low gain, tracking constant signals: (64)

USF: MIMO, linear, infinite-dimensional stabilization: (15)

Non-Minimum-Phase Systems, Tracking

SF: MIMO, tracking: (65,66)

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