VECTORS

A *vector* is a physical quantity that has the attributes of magnitude and direction, such as electric field intensity ($\boldsymbol{E}$), magnetic field intensity ($\boldsymbol{H}$), and current density ($\boldsymbol{J}$). The magnitude of a vector is always a nonnegative real number (such as 0, 2 or π). The *null vector* (or zero vector) $\boldsymbol{0}$ has a magnitude of zero, and has no direction. An ordinary number is called a *scalar*. A *unit vector* is a vector with a magnitude of 1 and any direction. Unit vectors are used to specify directions. Figure 1 shows several vectors of different magnitudes and directions.

The concept of a vector is a generalization of the concept of a displacement in space (such as "go 300 m east" or "move the capacitor 5 mm toward the connector end of the printed-circuit board"). Vectors that lie entirely in a single plane are called *two-dimensional* vectors, in contrast to *three-dimensional* vectors, which can point in any direction in three-dimensional space. A two- or three-dimensional vector is represented with an arrow, the length of which is proportional to the length of the vector, and the direction of which is the same as the direction of the vector. Abstract vectors with more than three dimensions are useful for computational purposes; see below.

The standard symbolic notations for vectors are boldface letters ($\boldsymbol{a}$) and lightface letters with an arrow overhead ($\vec{a}$).

Physically meaningful vectors have units. For example, the units of the electric field intensity vector $\boldsymbol{E}$ are volts per meter. The units of the vector are the same as the units of its magnitude. The set of all vectors with the same units is a *vector space*. Units are also called dimensions, but the term "dimension" has a different meaning in vector spaces.

Physical vectors such as position and momentum are properties of an object. The position vector of a point in space, $\boldsymbol{r}$, is defined by the object's x, y, and z coordinates: $\boldsymbol{r} = (x, y, z)$. However, electrical engineering also makes use of vector fields, such as the electric field, $\boldsymbol{E}$, the magnetic field, $\boldsymbol{H}$, and the current density field, $\boldsymbol{J}$. In general, a *vector field* is a function, for which we use the generic symbol $\boldsymbol{F}$, that assigns a vector $\boldsymbol{F}(\boldsymbol{r})$ to each point in space.

If a vector $\boldsymbol{a}$ with a magnitude a is multiplied by a number α, the result is a vector

$$\boldsymbol{b} = \alpha \boldsymbol{a}$$

in the same direction as $\boldsymbol{a}$ with a magnitude equal to αa. In mathematical language, a vector space is closed under *scalar multiplication.*

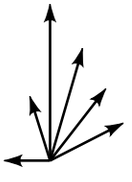

Figure 1. Several vectors of different magnitudes and directions.

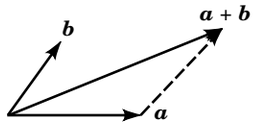

Figure 2. Head-to-tail method for combining successive displacements.

The rule for adding two vectors follows directly from the common-sense head-to-tail method for combining successive displacements, as shown in Fig. 2. The sum of two vectors is another vector; in mathematical terms, a vector space is closed under vector addition. Vector addition is *commutative,* that is, the sum of two vectors does not depend on the order in which the vectors are added:

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$$

Vector addition is also *associative,* meaning that the sum of three or more vectors does not depend on the manner in which the vectors are grouped:

$$\boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c}) = (\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c}$$

To compute the sum of two vectors, it often is convenient to express the vectors in terms of *components* along coordinate axes. The *Cartesian components* of a vector $\boldsymbol{a}$ are its components along the Cartesian x and y coordinate axes (and the z axis, in three dimensions). The component of a vector $\boldsymbol{a}$ along the x axis (for example) is usually written a_x. The x and y components of a two-dimensional vector are shown in Fig. 3, which also illustrates the important theorem that a vector is equal to the vector sum of its projections on the coordinate axes:

$$\boldsymbol{a} = a_x \hat{\boldsymbol{x}} + a_y \hat{\boldsymbol{y}}$$

From the Pythagorean theorem and from the fact that a_x and a_y can be thought of as distances along the x and y axes, which are mutually perpendicular, one sees that $a^2 = a_x^2 + a_y^2$. This gives the important formula

$$a = \sqrt{a_x^2 + a_y^2}$$

for the magnitude of a two-dimensional vector in terms of the magnitudes of the components along mutually perpendicular axes.

A two-dimensional vector $\boldsymbol{a}$ is the displacement vector from the origin to the point whose Cartesian coordinates are (a_x, a_y). For this reason a complex number (such as an impedance, or a phasor representing an alternating voltage or current) can be represented by a two-dimensional vector. For ex-

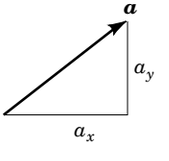

Figure 3. The x and y components of a 2-D vector.

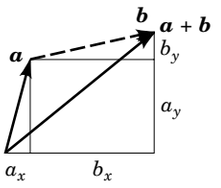

Figure 4. Illustration of components where sum of two or more vectors are equal to sums of components of the vectors.

ample, an impedance $\boldsymbol{Z} = 21 + j28\ \Omega$ can be represented by a two-dimensional vector with the Cartesian components (21, 28). The magnitude of the impedance in this example is $|\boldsymbol{Z}| = \sqrt{21^2 + 28^2} = 35\ \Omega$.

In three dimensions, the component resolution of a vector $\boldsymbol{a}$ is

$$\boldsymbol{a} = a_x\hat{\boldsymbol{x}} + a_y\hat{\boldsymbol{y}} + a_x\hat{\boldsymbol{z}}$$

and the formula for the vector's magnitude is

$$a = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

in terms of the Cartesian components a_x, a_y, and a_z.

Figure 4 illustrates the fact that the components of the sum of two or more vectors are equal to the sums of the components of the vectors:

$$\boldsymbol{a} + \boldsymbol{b} = (a_x\hat{\boldsymbol{x}} + a_y\hat{\boldsymbol{y}}) + (b_x\hat{\boldsymbol{x}} + b_y\hat{\boldsymbol{y}}) = (a_x + b_x)\hat{\boldsymbol{x}} + (a_y + b_y)\hat{\boldsymbol{y}}$$

In other words, the x component of $\boldsymbol{a} + \boldsymbol{b}$ is $a_x + b_x$.

The effective impedance of a combination of impedances in series can be found by vector addition. For example, if an impedance $\boldsymbol{Z}_1 = 1 + j3\ \Omega$ is in series with an impedance $\boldsymbol{Z}_2 = 5 - j4\ \Omega$, the effective impedance can be found by calculating the vector sum

$$\boldsymbol{Z} = \boldsymbol{Z}_1 + \boldsymbol{Z}_2 = (1, 3) + (5, -4) = (6, -1)$$

from which one sees that the effective impedance is $\boldsymbol{Z} = 6 - j1\ \Omega$. Figure 5 shows a graphical solution to this example.

To specify the direction of a two-dimensional vector numerically one uses the angle θ between the direction of the vector and a reference line, which usually is chosen as the positive x axis. For example, an impedance $\boldsymbol{Z} = 6 - j1\ \Omega$ (expressed in terms of its resistive and reactive components) can also be represented in terms of its magnitude and phase angle as $\boldsymbol{Z} = 6.08\angle -9.46°$ (Fig. 5).

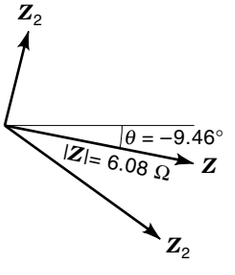

Figure 5. Graphical solution to $Z = 6 - j1\ \Omega$.

Given the x and y components of a two-dimensional vector, one can determine the angle θ that the vector makes with the x axis from the formula

$$\tan\theta = \frac{a_y}{a_x}$$

When using this formula numerically, one should remember that the arctangent function returns angles modulo 180°, that is, $\tan^{-1}(a_y/a_x)$ lies in the range (−90°, 90°), not in the range (0°, 360°) or (−180°, 180°). Therefore, one should determine the quadrant in which the angle lies by examining a_x and a_y, and adjust the value of the angle returned by the arctangent function by 180° or 360° if necessary. For example, if $a_x < 0$ and $a_y > 0$, then θ lies in the second quadrant, and therefore

$$\theta = \tan^{-1}\left(\frac{a_y}{a_x}\right) + 180°$$

in this case.

There are two meaningful products of two- or three-dimensional vectors. The *scalar product* (also called the dot product or inner product), which is written $\boldsymbol{a} \cdot \boldsymbol{b}$, is the number

$$\boldsymbol{a} \cdot \boldsymbol{b} = ab\cos\theta_{ab}$$

where θ_{ab} is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ (measured so that θ_{ab} is not less than 0° and is not greater than 180°). If $\hat{\boldsymbol{n}}$ is a unit vector, then $\hat{\boldsymbol{n}} \cdot \boldsymbol{a} = a\ \cos\ \theta_{na}$ is the component of $\boldsymbol{a}$ along the direction specified by $\hat{\boldsymbol{n}}$ (see Fig. 6).

Geometrically, $\boldsymbol{a} \cdot \boldsymbol{b}$ is equal to the magnitude of $\boldsymbol{a}$ times the magnitude of the component of $\boldsymbol{b}$ along the direction of $\boldsymbol{a}$. Trigonometric manipulation shows that the scalar product is equal to the sum of the products of the Cartesian components:

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_x b_x + a_y b_y + a_z b_z$$

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular to each other, then their scalar product vanishes:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{0}$$

and one says that $\boldsymbol{a}$ and $\boldsymbol{b}$ are *orthogonal.* One of the most important applications of the dot product is the computation of vector components. For example, to compute the rectangular components of a vector $\boldsymbol{a}$, one takes the dot product of $\boldsymbol{a}$ with the Cartesian unit vectors:

$$a_x = \hat{\boldsymbol{x}} \cdot \boldsymbol{a} \text{ and } a_y = \hat{\boldsymbol{y}} \cdot \boldsymbol{a}$$

The *vector product* (also called the cross product) of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is the vector $\boldsymbol{a} \times \boldsymbol{b}$ with the magnitude $ab|\sin\ \theta_{ab}|$,

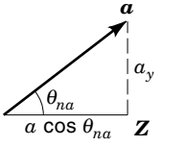

Figure 6. $\hat{\boldsymbol{n}} \cdot \boldsymbol{a} = a\ \cos\ \theta_{na}$ is the component of $\boldsymbol{a}$ along the direction specified by $\hat{\boldsymbol{n}}$.

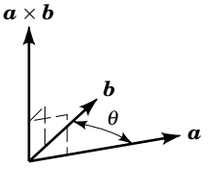

Figure 7. Right-hand rule.

orthogonal to both $\boldsymbol{a}$ and $\boldsymbol{b}$, and with the sense that one obtains by rotating $\boldsymbol{a}$ into $\boldsymbol{b}$ and using the right-hand rule (Fig. 7). To apply the right-hand rule, let the fingers of the right hand point from $\boldsymbol{a}$ to $\boldsymbol{b}$. The thumb then points in the direction of $\boldsymbol{a} \times \boldsymbol{b}$.

The magnitude of the vector product $\boldsymbol{a} \times \boldsymbol{b}$ is equal to the area of the parallelogram defined by $\boldsymbol{a}$ and $\boldsymbol{b}$. Referring to Fig. 8, one sees that the base of the parallelogram is the length of the vector $\boldsymbol{a}$, while the height is the length of $\boldsymbol{b}$ times the sine of the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. Hence the area of the parallelogram is equal to $ab\,|\sin\theta_{ab}|$, which is the magnitude of $\boldsymbol{a} \times \boldsymbol{b}$. Often one refers to $\boldsymbol{a} \times \boldsymbol{b}$ as the *directed area* of the parallelogram, because $\boldsymbol{a} \times \boldsymbol{b}$ has both magnitude and direction.

More generally, a directed area is any polygon, together with one of the two possible directions in space perpendicular to the polygon.

The *triple scalar product* $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$ is unchanged under cyclic permutations of $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, and changes sign under permutations that change the cyclic order:

$$\begin{aligned}\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) &= \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) \\ &= -\boldsymbol{a} \cdot (\boldsymbol{c} \times \boldsymbol{b}) - \boldsymbol{b} \cdot (\boldsymbol{a} \times \boldsymbol{c}) = -\boldsymbol{c} \cdot (\boldsymbol{b} \times \boldsymbol{a})\end{aligned}$$

The triple scalar product is equal to the volume of the parallelepiped whose edges are defined by the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, because the directed area of the base of the parallelepiped is $\boldsymbol{a} \times \boldsymbol{b}$, and the height of the parallelepiped is the component of $\boldsymbol{c}$ that is perpendicular to $\boldsymbol{a} \times \boldsymbol{b}$.

The value of the *triple vector product* $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$ is given by the *BAC–CAB* rule

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})$$

This formula is useful in electromagnetics and other areas in which one must express a vector $\boldsymbol{b}$ in terms of components that are parallel or perpendicular to a given unit vector $\hat{\boldsymbol{n}}$. For example, if one sets $\boldsymbol{a} = \boldsymbol{c} = \hat{\boldsymbol{n}}$ in the BAC–CAB rule, then one sees that

$$\boldsymbol{b} = (\hat{\boldsymbol{n}} \cdot \boldsymbol{b})\hat{\boldsymbol{n}} + \hat{\boldsymbol{n}} \times (\boldsymbol{b} \times \hat{\boldsymbol{n}})$$

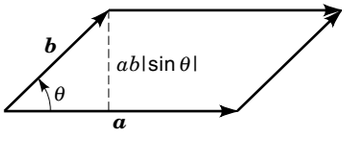

Figure 8. Base of parallelogram is length of vector $\boldsymbol{a}$; height is length of $\boldsymbol{b}$ times the sine of the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

where $\boldsymbol{b}$ is any vector. In this formula, the vector $(\hat{\boldsymbol{n}} \cdot \boldsymbol{b})\hat{\boldsymbol{n}}$ is parallel to $\hat{\boldsymbol{n}}$, and the vector $\hat{\boldsymbol{n}} \times (\boldsymbol{b} \times \hat{\boldsymbol{n}})$ is perpendicular to $\hat{\boldsymbol{n}}$.

For computational purposes, an *n-dimensional vector* is any array $v[\ ]$ with n elements $v[k] = v_k$, where $k = 0, \ldots, n - 1$. For example, a numerical solution of a differential equation such as

$$LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v = 0$$

is a vector, the elements $v[k] = v_k = v(t_k)$ of which are the approximate numerical values $v(t_k)$ of the function v at n sampling times $t_0, \ldots, t_{n-1}$. Vectors of time-sampled values are examples of abstract vectors, that is, objects that do belong to three-dimensional space, but that have many of the properties of physical vectors.

An *abstract vector space* V is a set, the elements $\bar{x}, \bar{y}, \bar{z}, \ldots$ of which obey the following axioms, where $\alpha, \beta, \ldots$ are scalars:

1. An operation of addition is defined, such that, for every pair of vectors $\bar{x}, \bar{y}$ in V, the vector sum $\bar{x} + \bar{y}$ belongs to V, and $\bar{y} + \bar{x} = \bar{x} + \bar{y}$.
2. Vector addition is associative: $\bar{x} + (\bar{y} + \bar{z}) = (\bar{x} + \bar{y}) + \bar{z}$.
3. A null vector $\bar{0}$ exists, with the property that for every vector $\bar{x}$ in V, $\bar{x} + \bar{0} = \bar{x}$..
4. For every vector $\bar{x}$ in V, there exists a vector $-\bar{x}$ such that $\bar{x} + (-\bar{x}) = \bar{x} - \bar{x} = \bar{0}$.
5. For every vector $\bar{x}$ in V, and for every scalar α, there is a scalar multiple $\alpha\bar{x}$ that belongs to V.
6. Scalar multiplication obeys the distributive laws $\alpha(\bar{x} + \bar{y}) = \alpha\bar{x} + \alpha\bar{y}$ and $(\alpha + \beta)\bar{x} = \alpha\bar{x} + \beta\bar{x}$.
7. Scalar multiplication is associative, meaning that $\alpha(\beta\bar{x}) = (\alpha\beta)\bar{x}$.
8. Multiplying any vector $\bar{x}$ by the zero scalar 0 gives the null vector $\bar{0}$.
9. Multiplying any vector $\bar{x}$ by the unit scalar 1 gives back $\bar{x}$: $1\bar{x} = \bar{x}$.

The concept of an abstract vector space is very widely applicable. The "vectors" $\bar{x}$ can be functions, column vectors, row vectors, or even matrices. Any property that is true for abstract vectors is true for functions, column vectors, and so on.

A set of abstract vectors $\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{m-1}$ is called *linearly independent* if the only scalars $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ such that

$$\sum_{k=0}^{m-1} \alpha_k \bar{x}_k = \bar{0}$$

are $\alpha_0 = \alpha_1 = \cdots = \alpha_{m-1} = 0$. On the other hand, if this equation holds with some nonzero values of the scalars $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$, then the vectors $\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_{m-1}$ are called *linearly dependent.* In particular, if two abstract vectors $\bar{x}, \bar{y}$ are linearly dependent, then there is a number α such that $\bar{y} = \alpha\bar{x}$. For example, the functions $\sin \omega t$ and $\cos \omega t$ are linearly independent, for there is no number α such that $\cos \omega t = \alpha \sin \omega t$.

It can be proved that, for any given abstract vector space V, the number of linearly independent vectors in a set $\bar{x}_0, \bar{x}_1,$

. . ., x_{m-1} is less than or equal to a certain maximum number n, which is called the *dimension* of the vector space. In effect, the dimension of a vector space is equal to the number of degrees of freedom in the space. For example, there are at most three linearly independent vectors in any set of three-dimensional vectors, because every three-dimensional vector is linearly dependent on the Cartesian unit vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$. A set of linearly independent vectors $\overline{e}_0$, $\overline{e}_1$, . . ., e_{n-1} that has the property that every other vector in a vector space V can be expressed as a linear combination

$$\overline{v} = \sum_{k=0}^{m-1} v_k \overline{e}_k$$

is called a basis of V. If the basis vectors are normalized so that each one has unit length and they are mutually orthogonal, then the basis is called *orthonormal.* The Cartesian unit vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ are an orthonormal basis of the space of three-dimensional vectors. Orthonormal bases of vector spaces whose elements are functions are used in many applications, ranging from electromagnetic boundary-value problems to the theory of random processes.

VECTOR COMPONENTS IN ORTHOGONAL CURVILINEAR COORDINATE SYSTEMS

Some electromagnetic engineering problems, such as scattering, radiation by an antenna, and wave propagation in cylindrical waveguides, take their simplest form when expressed in coordinates that are adapted to the geometrical symmetry of the problem. For example, an optical fiber can be idealized as a cylinder. The natural coordinate system in this case is the cylindrical system. The *circular cylindrical coordinates* of a point $\boldsymbol{r} = (x, y, z)$ in three-dimensional space are (ρ, ϕ, z), where

$$\rho = \sqrt{x^2 + y^2}$$

is the perpendicular distance from the z axis to the point $\mathbf{r}$,

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

is the azimuthal angle, measured from the X axis to the line in the XY plane that goes from the origin to the point $(x, y, 0)$, and z is equal to the Z Cartesian coordinate of $\boldsymbol{r}$, as is shown in Fig. 9. Evidently cylindrical coordinates are simply

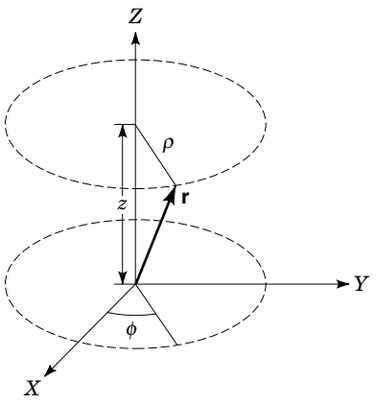

Figure 9. Circular cylindrical coordinates of point $\mathbf{r}$.

plane polar coordinates (ρ, ϕ), with a Z coordinate added to describe three-dimensional objects. The other important orthogonal curvilinear coordinates for engineering applications are spherical polar coordinates, which we discuss below.

One of the natural unit vectors in circular cylindrical coordinates is $\hat{\boldsymbol{z}}$, which points in the direction that one takes if one increases z while holding x and y (or ρ and ϕ) constant. Another is the unit vector $\hat{\boldsymbol{\rho}}$ that points in the direction in which one moves if one increases ρ while holding z and ϕ constant. Still another is the unit vector $\hat{\boldsymbol{\phi}}$ that points in the direction of increasing ϕ (with z and ρ held constant). In terms of the Cartesian unit vectors $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$, the circular cylindrical unit vectors in the XY plane (which are the same as the plane polar unit vectors) are

$$\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{x}} \cos\phi + \hat{\boldsymbol{y}} \sin\phi$$

and

$$\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}} \sin\phi + \hat{\boldsymbol{y}} \cos\phi$$

Because $\hat{\boldsymbol{z}}$ is orthogonal to $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{y}}$,

$$\hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{\rho}} = 0 \text{ and } \hat{\boldsymbol{z}} \cdot \hat{\boldsymbol{\phi}} = 0$$

Substituting for $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ from the formulas just above, one finds that

$$\hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\rho}} = 0$$

These equations demonstrate that $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\boldsymbol{z}}$ form an orthonormal system of unit vectors, which one can use (instead of the fixed rectangular unit vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$) to express any vector in terms of its ρ, ϕ, and z components. The unit vectors $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\boldsymbol{z}}$ define a local Cartesian coordinate system, the axes of which point in different directions, depending on the location of the point $\boldsymbol{r}$. Because the unit vectors are mutually orthogonal, circular cylindrical coordinates are an example of orthogonal curvilinear coordinates.

Any vector field $\boldsymbol{F}$ can be expressed in terms of its components along $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\boldsymbol{z}}$ as follows:

$$\mathbf{F} = F_\rho \hat{\boldsymbol{\rho}} + F_\phi \hat{\boldsymbol{\phi}} + F_z \hat{\boldsymbol{z}}$$

To compute the circular cylindrical components of $\boldsymbol{F}$, one takes the dot product with the circular cylindrical unit vectors, obtaining

$$F_\rho = \hat{\boldsymbol{\rho}} \cdot \boldsymbol{F} = F_x \cos\phi + F_y \sin\phi$$

and

$$F_\phi = \hat{\boldsymbol{\phi}} \cdot \boldsymbol{F} = -F_x \sin\phi + F_y \cos\phi$$

These formulas are used to compute the circular cylindrical components, given the rectangular components.

The formulas that give the rectangular (XYZ) unit vectors in terms of the circular cylindrical (or plane polar) unit vectors are

$$\hat{\boldsymbol{x}} = \hat{\boldsymbol{\rho}} \cos\phi - \hat{\boldsymbol{\phi}} \sin\phi$$

and

$$\hat{\boldsymbol{y}} = \hat{\boldsymbol{\rho}} \sin\phi + \hat{\boldsymbol{\phi}} \cos\phi$$

The unit vector $\hat{\boldsymbol{z}}$ is the same in both coordinate systems. These formulas can be used to compute the rectangular components, given the circular cylindrical components.

In general, if a global coordinate system is not Cartesian, it would be convenient to be able to "attach" a local Cartesian coordinate system at each point $\boldsymbol{r}$. Curvilinear coordinate systems in which this is possible are called *orthogonal curvilinear coordinates*. Examples besides circular cylindrical and spherical polar coordinates include elliptic and parabolic coordinates in two dimensions and, in three dimensions, elliptic cylinder coordinates, parabolic cylinder coordinates, conical coordinates, parabolic coordinates, prolate spheroidal coordinates, oblate spheroidal coordinates, ellipsoidal coordinates and paraboloidal coordinates.

The *spherical polar coordinates* of a point $\boldsymbol{r} = (x, y, z)$ are (r, θ, ϕ), where

$$r = \sqrt{x^2 + y^2 + z^2}$$

is the distance from the origin to the point $\boldsymbol{r}$,

$$\theta = \cos^{-1}\left(\frac{z}{r}\right)$$

is the polar angle, measured from the Z axis to the line that goes from the origin to $\boldsymbol{r}$, and

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

is the azimuthal angle, measured from the X axis to the line in the XY plane that goes from the origin to the point $(x, y, 0)$, as is shown in Fig. 10. To locate points on the surface of a sphere (defined by the equation r = constant), one needs only the polar angle θ and the azimuthal angle ϕ.

The formulas for the spherical polar unit vectors, $\hat{\boldsymbol{r}}$, $\hat{\theta}$, and $\hat{\boldsymbol{\phi}}$ in terms of the rectangular unit vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ are

$$\hat{\boldsymbol{r}} = \hat{\boldsymbol{x}} \sin\theta \cos\phi + \hat{\boldsymbol{y}} \sin\theta \sin\phi + \hat{\boldsymbol{z}} \cos\theta$$
$$\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{x}} \cos\theta \cos\phi + \hat{\boldsymbol{y}} \cos\theta \sin\phi - \hat{\boldsymbol{z}} \sin\theta$$
$$\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}} \sin\phi + \hat{\boldsymbol{y}} \cos\phi$$

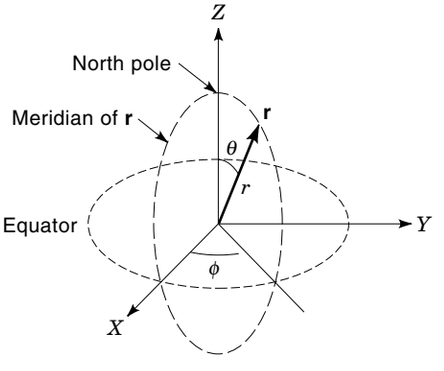

Figure 10. Spherical polar coordinates of point **r**.

These formulas imply that $\hat{\boldsymbol{r}}$, $\hat{\theta}$, and $\hat{\boldsymbol{\phi}}$ are mutually orthogonal,

$$\hat{\boldsymbol{r}} \cdot \hat{\theta} = 0$$
$$\hat{\boldsymbol{r}} \cdot \hat{\phi} = 0$$
$$\hat{\boldsymbol{\theta}} \cdot \hat{\phi} = 0$$

and therefore define a set of local Cartesian coordinate axes that are attached at the point $\boldsymbol{r}$. The inverse relations

$$\hat{\boldsymbol{x}} = \hat{\boldsymbol{r}} \sin\theta \cos\phi + \hat{\theta} \cos\theta \cos\phi - \hat{\phi} \sin\theta$$
$$\hat{\boldsymbol{y}} = \hat{\boldsymbol{r}} \sin\theta \sin\phi + \hat{\theta} \cos\theta \sin\phi + \hat{\phi} \sin\theta$$
$$\hat{\boldsymbol{z}} = \hat{\boldsymbol{r}} \cos\theta - \hat{\theta} \sin\theta$$

give the spherical polar unit vectors in terms of the rectangular unit vectors.

To compute the spherical polar coordinates of a vector field $\boldsymbol{F}$ whose rectangular components are known, take the dot products of the spherical polar unit vectors with $\boldsymbol{F}$:

$$F_r = F_x \sin\theta \cos\phi + F_y \sin\theta \sin\phi + F_z \cos\theta$$
$$F_\theta = F_x \cos\theta \cos\phi + F_y \cos\theta \sin\phi - F_z \sin\theta$$
$$F_\phi = -F_x \sin\phi + F_y \cos\phi$$

To compute the rectangular components given the spherical polar coordinates, use the formulas given above for $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ in terms of the spherical polar unit vectors $\hat{\boldsymbol{r}}$, $\hat{\theta}$, and $\hat{\boldsymbol{\phi}}$.

VECTOR INTEGRATION

Integrals that involve vectors are important in electrical engineering because important quantities such as the electrical current through a surface or the work done in moving a charge along a three-dimensional path depend upon the magnitude and direction of a vector, as well as the orientation of a surface or line.

The *line integral* of a vector field $\boldsymbol{F}$ along a curve C is defined as the limit of the sum of contributions from line segments that approximate the curve:

$$\int_C \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{l} = \lim_{N\to\infty} \sum_{i=0}^{N-1} \boldsymbol{F}(\boldsymbol{r}_i) \cdot (\boldsymbol{r}_{i+1} - \boldsymbol{r}_i)$$

For example, this formula gives the work that is done by a force field F acting on a particle as it moves along the curve C. The potential difference V_{AB} between two points A and B is equal to minus the line integral of the electric field intensity $\boldsymbol{E}$ from A to B,

$$V_{AB} = -\int_A^B \boldsymbol{E} \cdot d\boldsymbol{l}$$

In this example, the value of the potential difference is independent of the path taken from A to B. In general, however, the value of a line integral depends upon the path. For exam-

ple, the electromotive force $\mathscr{E}$ around a wire loop L that is immersed in a time-varying magnetic field,

$$\mathscr{E} = \oint_L \boldsymbol{E} \cdot d\boldsymbol{l}$$

depends upon the loop's size, shape, and number of turns.

The *surface integral* of a vector field $\boldsymbol{F}$ over a surface S is the limit of the sum, over plane polygons (with directed areas $\Delta \boldsymbol{S}_i$) that approximate S, of products of the form (area of $\Delta \boldsymbol{S}_i$ times component of $\boldsymbol{F}$ perpendicular to $\Delta \boldsymbol{S}_i$):

$$\int_S \boldsymbol{F}(\boldsymbol{r}) \cdot d\boldsymbol{S} = \lim_{N\to\infty} \sum_{i=0}^{N-1} \boldsymbol{F}(\boldsymbol{r}_i) \cdot \Delta \boldsymbol{S}_i$$

For example, the electric current I_S that flows through a surface S is equal to the surface integral of the current density $\boldsymbol{J}$ over S,

$$I_S = \int_S \boldsymbol{J}(\boldsymbol{r}) \cdot d\boldsymbol{S}$$

Gauss' law states that the electric flux through a closed surface, $\int_S \boldsymbol{D} \cdot d\boldsymbol{S}$, is equal to the charge enclosed in S.

Several important formulas involve the integral of a scalar field ψ, which is defined as the limit of a sum, over small volumes ΔV_i that approximate V, of products of the form (value of ψ times volume ΔV_i):

$$\int_V \psi(\boldsymbol{r}) dV = \lim_{N\to\infty} \sum_{i=0}^{N-1} \psi(\boldsymbol{r}_i) \Delta V_i$$

For example, the total electric charge distributed over a volume V is equal to the volume integral of the electric charge density ρ.

VECTOR OPERATORS

Electromagnetic engineering makes intensive use of the gradient, divergence, curl, and Laplacian, which are called *vector differential operators* because they either operate on vectors, produce a vector result, or both.

The *gradient* of a scalar field ψ (written as $\nabla\psi$, or sometimes as gradψ) is a vector field that, at each location $\boldsymbol{r}$ in space, points in the direction in which ψ increases most rapidly. The magnitude of $\nabla\psi$ is equal to the maximum rate at which ψ changes. These properties are evident from the definition of the gradient in terms of its component along an arbitrary unit vector $\hat{\boldsymbol{n}}$:

$$\hat{\boldsymbol{n}} \cdot (\nabla\psi(\boldsymbol{r})) = \lim_{\Delta l\to 0} \frac{\psi(\boldsymbol{r} + \hat{\boldsymbol{n}}\Delta l) - \psi(\boldsymbol{r})}{\Delta l}$$

The change in ψ under a small displacement from $\boldsymbol{r}$ to $\boldsymbol{r} + \hat{\boldsymbol{n}}\Delta l$ is approximately equal to the dot product of $\hat{\boldsymbol{n}}\Delta l$ with $\nabla\psi$:

$$\psi(\boldsymbol{r} + \hat{\boldsymbol{n}}\Delta l) - \psi(\boldsymbol{r}) \approx (\nabla\psi(\boldsymbol{r})) \cdot (\hat{\boldsymbol{n}}\Delta l)$$

For example, if ψ is the electric potential, then $\Delta\psi$ is the work done by an external force in moving a unit charge from $\boldsymbol{r}$ to $\boldsymbol{r} + \hat{\boldsymbol{n}}\Delta l$. The equation $\boldsymbol{E} = -\nabla\psi$ implies that the electric field intensity is equal to minus the gradient of the electric potential. The *gradient theorem* states that the line integral from a point $\boldsymbol{r}$ to a point $\boldsymbol{s}$ of the gradient of a scalar field ψ is equal to the difference of the values of ψ at $\boldsymbol{s}$ and $\boldsymbol{r}$:

$$\int_r^s (\nabla\psi) \cdot d\boldsymbol{l} = \psi(\boldsymbol{s}) - \psi(\boldsymbol{r})$$

A special case is the familiar formula for the integral of an ordinary derivative,

$$\int_a^b \frac{d\psi}{dx} dx = \psi(b) - \psi(a)$$

An extremely important consequence of the gradient theorem is the fact that the integral of a gradient is independent of the path of integration.

Formulas for the basic vector operators in the three most important coordinate systems (rectangular, circular cylindrical and spherical polar coordinates) are needed constantly in electromagnetics. The gradient of a scalar field ψ is

$$\nabla\psi = \frac{\partial\psi}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial\psi}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial\psi}{\partial z}\hat{\boldsymbol{z}}$$

in Cartesian coordinates,

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{\phi} + \frac{\partial\psi}{\partial z}\hat{\boldsymbol{z}}$$

in circular cylindrical coordinates, and

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\phi}$$

in spherical polar coordinates.

The *divergence* of a vector field $\boldsymbol{F}$ (written as $\nabla \cdot \boldsymbol{F}$, or sometimes as div$\boldsymbol{F}$) is defined as the limit, as the volume ΔV approaches zero, of the integral of $\boldsymbol{F}$ over the surface ΔS that bounds ΔV, divided by ΔV:

$$\nabla \cdot \boldsymbol{F} = \lim_{\Delta V\to 0} \frac{\int_{\Delta S} \boldsymbol{F} \cdot d\boldsymbol{S}}{\Delta V}$$

If $\boldsymbol{F}$ is a flux vector (a rate of transfer of some physical quantity Q per unit area, per unit time), then the divergence $\nabla \cdot \boldsymbol{F}$ is equal to the rate (per unit volume) at which Q is created or destroyed. In other words, the divergence of $\boldsymbol{F}$ is nonzero wherever there are sources or sinks of the physical quantity Q of which $\boldsymbol{F}$ is the flux. *Gauss' theorem,*

$$\int_V (\nabla \cdot \boldsymbol{F})\, dV = \int_S \boldsymbol{F} \cdot d\boldsymbol{S}$$

states that the integral of the divergence of $\boldsymbol{F}$ over any volume V is equal to the integral of $\boldsymbol{F}$ over the surface S that bounds V. For example, according to Gauss' theorem and one of Maxwell's equations,

$$\int_S \boldsymbol{D} \cdot d\boldsymbol{S} = \int_V \nabla \cdot \boldsymbol{D} dV = \int_V \rho dV = Q_V$$

where $\boldsymbol{D}$ is the electric flux density, ρ is the electric charge density, and Q_V is the electric charge enclosed in V. This is simply a restatement of Gauss' law. The conservation of electric charge can be expressed in the equation

$$\nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0$$

in terms of the divergence of the electric current density $\boldsymbol{J}$.

Explicit formulas for the divergence $\nabla \cdot \boldsymbol{F}$ of a vector field $\boldsymbol{F}$ are

$$\nabla \cdot \boldsymbol{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

in Cartesian coordinates,

$$\nabla \cdot \boldsymbol{F} = \frac{\partial (\rho F_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

in circular cylindrical coordinates, and

$$\nabla \cdot \boldsymbol{F} = \frac{1}{r^2}\frac{\partial (r^2 F_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (\sin\theta F_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}$$

in spherical polar coordinates.

The *curl* of a vector field $\boldsymbol{F}$ (written as $\nabla \times \boldsymbol{F}$, or sometimes as curl$\boldsymbol{F}$) is defined as the limit, as the area of a small, flat surface $\Delta \boldsymbol{S}$ ($\perp \hat{\boldsymbol{n}}$) approaches zero, of the line integral of $\boldsymbol{F}$ around the closed curve ΔC that bounds $\Delta \boldsymbol{S}$:

$$\hat{\boldsymbol{n}} \cdot (\nabla \times \boldsymbol{F}) = \lim_{\Delta S \to 0} \frac{\oint_{\Delta C} \boldsymbol{F} \cdot d\boldsymbol{l}}{\Delta S}$$

Stokes' theorem,

$$\int_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S} = \oint_C \boldsymbol{F} \cdot d\boldsymbol{l}$$

states that the line integral of a vector field $\boldsymbol{F}$ around a closed curve C is equal to the integral of the curl of $\boldsymbol{F}$ over any surface enclosed by C. Stokes' theorem and Faraday's law,

$$\mathscr{E} = \oint_C \boldsymbol{E} \cdot d\boldsymbol{l} = -\frac{\partial \Phi_C}{\partial t}$$

(where $\Phi_C = \int_S \boldsymbol{B} \cdot d\boldsymbol{S}$ is the magnetic flux enclosed by C, and $\boldsymbol{B}$ is the magnetic induction), imply one of Maxwell's equations,

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

In a fluid, the curl of the velocity field $\boldsymbol{v}$ is nonzero only where there is a net circulation of fluid, as there is in a whirlpool or eddy. In a region of space, or in a material, in which there are no conduction currents, the electric field intensity $\boldsymbol{E}$ obeys the wave equation

$$\nabla \times (\nabla \times \boldsymbol{E}) + \mu\epsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \boldsymbol{0}$$

The curl, gradient and divergence obey the operator identities

$$\nabla \times (\nabla \psi) = \boldsymbol{0}$$

(the curl of a gradient is zero) and

$$\nabla \cdot (\nabla \times \boldsymbol{F}) = 0$$

(the divergence of a curl is zero). The relation $\nabla \times (\nabla \psi) = \boldsymbol{0}$ implies that the integral of a gradient around a closed path vanishes, and the relation $\nabla \cdot (\nabla \times \boldsymbol{F}) = 0$ implies that the curl of a vector field cannot describe sources or sinks.

Formulas for the curl $\nabla \times \boldsymbol{F}$ of a vector field $\boldsymbol{F}$ in the major coordinate systems are

$$\nabla \times \boldsymbol{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\boldsymbol{z}}$$

in Cartesian coordinates,

$$\nabla \times \boldsymbol{F} = \left(\frac{1}{\rho}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}\right)\hat{\boldsymbol{\phi}} + \frac{1}{\rho}\left(\frac{\partial (\rho F_\phi)}{\partial \rho} - \frac{\partial F_\rho}{\partial \phi}\right)\hat{\boldsymbol{z}}$$

in circular cylindrical coordinates, and

$$\nabla \times \boldsymbol{F} = \frac{1}{r\sin\theta}\left(\frac{\partial (\sin\theta F_\phi)}{\partial \theta} - \frac{\partial F_\theta}{\partial \phi}\right)\hat{\boldsymbol{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial \phi} - \frac{\partial (r F_\phi)}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial (r F_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta}\right)\hat{\phi}$$

in spherical polar coordinates.

The *Laplacian* operator on a scalar field ψ (usually written as $\nabla^2 \psi$) is defined as the divergence of the gradient:

$$\nabla^2 \psi = \nabla \cdot (\nabla \psi)$$

The Laplacian of a scalar field ψ is nonzero only at points at which there are sources or sinks of the vector field $\nabla \psi$. For example, the right-hand side of Poisson's equation,

$$\nabla^2 \psi = -\frac{\rho}{\epsilon_0}$$

is nonzero only where the electric charge density ρ is nonzero.

In Cartesian coordinates, the formula for the Laplacian is

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}$$

The formulas for the Laplacian in the most important orthogonal curvilinear coordinate systems are

$$\nabla^2 \psi = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}$$

in circular cylindrical coordinates, and

$$\nabla^2 \psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2}$$

in spherical polar coordinates.

The Laplacian of a vector field is defined in such a way that the equation

$$\nabla \times (\nabla \times \boldsymbol{F}) = \nabla(\nabla \cdot \boldsymbol{F}) - \nabla^2 \boldsymbol{F}$$

is true in all orthogonal curvilinear coordinate systems. In Cartesian coordinates, the Laplacian acts only on the components of a vector field $\boldsymbol{F}$:

$$\nabla^2 \boldsymbol{F} = (\nabla^2 F_x)\hat{\boldsymbol{x}} + (\nabla^2 F_y)\hat{\boldsymbol{y}} + (\nabla^2 F_z)\hat{\boldsymbol{z}}$$

However, in curvilinear coordinates (such as circular cylindrical or spherical polar coordinates), the gradient and divergence operators act on the unit vectors as well as on the components of $\boldsymbol{F}$, because the unit vectors depend upon the position vector, $\boldsymbol{r}$.

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VECTOR SPACES, LINEAR. See LINEAR ALGEBRA.