WALSH FUNCTIONS

Orthogonal properties (1,2) of the familiar sine–cosine functions have been known for over two centuries. Use of such functions in an elegant manner to solve complex analytical problems was initiated by the work of the famous mathematician Baron Jean-Baptiste-Joseph Fourier (3). The system of sine and cosine functions plays a distinguished role in many areas of electrical engineering. There are a number of historical and practical reasons for this. From the theoretical point of view, one of the major reasons is that Fourier series and Fourier transform permit the representation of a large class of functions by a superposition of sine and cosine functions. This representation makes it possible to apply the concept of frequency, which was originally defined for sine and cosine only, to other functions.

In the fields of circuit analysis, control theory, and communications the complete and orthogonal properties of sine and cosine functions produce attractive solutions. But with the application of digital techniques and semiconductor technology in these areas, awareness for other more general complete systems of orthogonal functions has developed. This class of functions, though not possessing some of the desirable properties of sine-cosine functions in linear time-invariant networks, has other advantages rendering its use more directly applicable to all such applications in the context of digital technology. Many members of this class of orthogonal functions are piecewise constant basis functions (PCBF), thus resembling the high-low switching characteristic of semiconductor devices. Walsh functions belong to the class of PCBFs that have been developed in the twentieth century and have played an important role in scientific and engineering applications. The mathematical techniques of studying functions, signals, and systems through series expansions in orthogonal complete sets of basis functions are now a standard tool in all branches of science and engineering. Actually, the signals involved in Morse telegraphy are PCBFs, but no mathematical study of these signals was made prior to the beginning of the twentieth century.

The origin of the mathematical study of PCBFs is due to the Hungarian mathematician Alfred Haar (studies completed 1910–1912), who used a set of functions now bearing his name. These functions have not found much use in comparison to the Walsh and block-pulse functions. The development and utilization of Walsh functions has been strongly influenced by the parallel developments in digital electronics and computer science and engineering. Efforts to replace Fourier transforms by Walsh-type transforms have been made in communication, signal processing, image processing, pattern recognition, and so forth. Applications of Walsh functions in the systems and control field were begun only about two decades ago, and developments since then have occurred rapidly.

DIFFERENT TYPES OF PIECEWISE CONSTANT BASIS FUNCTION

Haar Functions

The set of Haar functions is periodic, orthogonal, and complete and was proposed in 1910 by Alfred Haar (4). Figure 1 shows the set of first eight Haar functions. A recurrence relation that enables one to generate the Haar functions $\{\text{har}(j, n, t)\}$ in the semi-open interval $t \in [0, 1)$ is given by (5). The first member of the set is

$$\text{har}(0, 0, t) = 1 \qquad t \in [0, 1)$$

while the general term for other members is given by

$$\text{har}(j, n, t) = \begin{cases} 2^{j/2} & \dfrac{n-1}{2^j} \le t < \dfrac{n-0.5}{2^j} \\ -2^{j/2} & \dfrac{n-0.5}{2^j} \le t < \dfrac{n}{2^j} \\ 0 & \text{elsewhere} \end{cases}$$

where j, n, and m are integers governed by the relation

$$0 \le j \le \log_2 m \qquad 1 \le n \le 2^j$$

The number of members in the set is of the form $m = 2^k$, k being a positive integer.

Haar's set is such that the formal expansion of a given continuous function in terms of Haar functions converges uniformly to the given function (6).

Rademacher Functions

Rademacher functions are an incomplete set of orthonormal functions which were developed by the German mathemati-

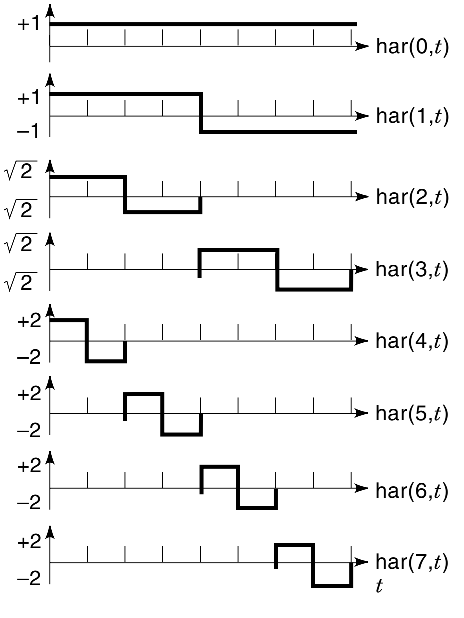

Figure 1. A set of the first eight Haar functions.

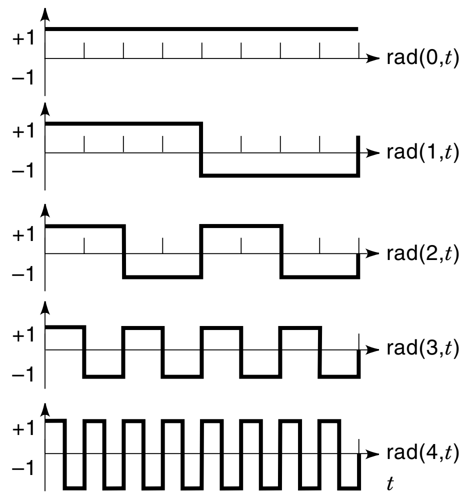

Figure 2. A set of the first five Rademacher functions.

cian H. Rademacher in 1922 (7). Figure 2 shows the set of the first five Rademacher functions. The Rademacher function of index m, denoted by rad(m, t) is given by a square wave of unit amplitude and 2^{m-1} cycle in the semi-open interval [0, 1), with the exception of rad(0, t) which has a constant value of unity throughout the interval. Rademacher functions can be generated using the recurrence relation (8)

$$\mathrm{rad}(m,t) = \mathrm{rad}(1, 2^{m-1}t), \qquad m \neq 0$$

with

$$\mathrm{rad}(1,t) = \begin{cases} 1 & t \in [0, 0.5) \\ -1 & t \in [0.5, 1) \end{cases}$$

Walsh Functions

The incomplete set of Rademacher functions was completed by J. L. Walsh in 1923, to form the complete orthogonal set of rectangular functions we now call the Walsh functions (9).

As indicated by Walsh, there are many possible orthogonal function sets of this kind. Since Walsh's work several researchers have suggested orthogonal series formed with the help of combinations of the well-known piecewise constant orthogonal functions (10–12).

In his original paper Walsh pointed out that, ". . . Haar's set is, however, merely one of an infinity of sets which can be constructed of functions of this same character." While proposing his new set of orthogonal functions, Walsh wrote, ". . . each function takes only the values +1 and −1 except at a finite number of points of discontinuity, where it takes the value zero."

It is interesting to note that some of the square wave patterns of individual Walsh functions appear in several ancient designs (13). Chess board or checker board designs are two-dimensional Walsh functions, whereas the Rubik Cube is a three-dimensional Walsh function.

The set of Walsh functions is generally classified into three groups. These groups differ from one another in that the order in which individual functions appear is different. The three types of orderings are: (1) Sequency or Walsh ordering, (2) Dyadic or Paley ordering, and, (3) Natural or Hadamard ordering. In what follows, we discuss some aspects of each of these orderings.

Sequency or Walsh Ordering

This is the ordering which was originally employed by Walsh (9). Sequency ordered Walsh functions are arranged in ascending order of zero crossings. Sequency is defined as one-half the average number of zero crossings over the unit interval [0, 1), and is used as a measure of generalized frequency of wave forms. Figure 3 shows a set of the first eight sequency order Walsh functions wal(m, t), where m is the sequency order number and $0 \leq t < 1$.

If each waveform is divided into eight intervals, the magnitude of the waveform can be expressed as a matrix

$$W(m,l) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix} \qquad (1)$$

where m denotes the order of Walsh function (the row of the matrix), l the corresponding bit of this order (the column of the same matrix), and $W(m, l)$ is called the Walsh matrix.

Walsh functions are either symmetrical or asymmetrical with respect to their middle point. They are called *cal* and *sal* functions respectively. These functions are expressed as

$$\mathrm{wal}(2m,t) = \mathrm{cal}(m,t) \qquad m = 1, 2, \ldots, \frac{N}{2} \qquad (2)$$

$$\mathrm{wal}(2m-1,t) = \mathrm{sal}(m,t) \qquad m = 1, 2, \ldots, \frac{N}{2} \qquad (3)$$

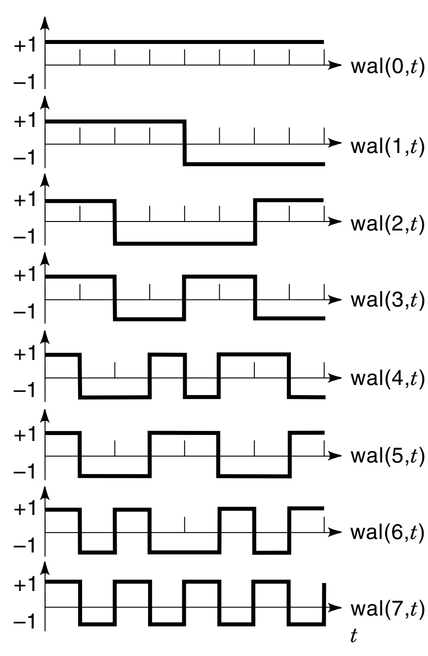

Figure 3. A set of the first eight Walsh functions arranged in sequency order.

Because of their symmetrical characteristic, sal and cal terms can be thought of as being analogous to the sine and cosine terms of the Fourier series.

Similarly to the Fourier series representation, the Walsh series representation of a time function that is absolutely integrable in [0, 1) is defined as

$$f(t) = \sum_{m=0}^{\infty} F_m \mathrm{wal}(m, t) \tag{4}$$

where F_m is the coefficient of the Walsh function of $f(t)$. It is desirable to determine the coefficient such that the integral square error is minimized

$$\epsilon = \int_0^1 \left[f(t) - \sum_{m=0}^{\infty} F_m \mathrm{wal}(m, t) \right]^2 dt$$

Taking the partial derivative of ϵ with respect to F_m and setting it equal to zero yields

$$F_m = \int_0^1 f(t)\mathrm{wal}(m, t)\, dt \qquad m = 0, 1, 2, \ldots \tag{5}$$

This simple result is due to the orthonormal property of Walsh functions. Let us illustrate the Walsh series expansion for the ramp function

$$f(t) = t$$

Substituting $f(t)$ into Eq. (5) and taking four terms yields

$$F_0 = \int_0^1 t\mathrm{wal}(0, t)\, dt = \frac{1}{2}$$
$$F_1 = \int_0^1 t\mathrm{wal}(1, t)\, dt = -\frac{1}{4}$$
$$F_2 = \int_0^1 t\mathrm{wal}(2, t)\, dt = 0$$
$$F_3 = \int_0^1 t\mathrm{wal}(3, t)\, dt = -\frac{1}{8}$$

After substituting these obtained values of coefficients into Eq. (4) we have

$$t = \frac{1}{2}\mathrm{wal}(0, t) - \frac{1}{4}\mathrm{wal}(1, t) - \frac{1}{8}\mathrm{wal}(3, t)$$

which is the four term sequency ordered Walsh function series expansion of the ramp function.

Dyadic or Paley Ordering

The dyadic type of ordering was introduced by Paley (14). The dyadic order is obtained by generating Walsh functions from successive Rademacher functions. The set of Walsh and Rademacher functions that are referred to here as pal(n, t) and rad(q, t) respectively have the following relation:

$$\mathrm{pal}(0, t) = \mathrm{rad}(0, t)$$
$$\mathrm{pal}(1, t) = \mathrm{rad}(1, t)$$
$$\mathrm{pal}(2, t) = [\mathrm{rad}(2, t)]^1[\mathrm{rad}(1, t)]^0$$
$$\mathrm{rad}(3, t) = [\mathrm{rad}(2, t)]^1[\mathrm{rad}(1, t)]^1$$
$$\mathrm{pal}(4, t) = [\mathrm{rad}(3, t)]^1[\mathrm{rad}(2, t)]^0[\mathrm{rad}(1, t)]^0$$
$$\mathrm{pal}(5, t) = [\mathrm{rad}(3, t)]^1[\mathrm{rad}(2, t)]^0[\mathrm{rad}(1, t)]^1$$
$$\mathrm{pal}(6, t) = [\mathrm{rad}(3, t)]^1[\mathrm{rad}(2, t)]^1[\mathrm{rad}(1, t)]^0$$
$$\mathrm{pal}(7, t) = [\mathrm{rad}(3, t)]^1[\mathrm{rad}(2, t)]^1[\mathrm{rad}(1, t)]^1$$
$$\vdots$$
$$\mathrm{pal}(n, t) = [\mathrm{rad}(q, t)]^{b_q}[\mathrm{rad}(q-1, t)]^{b_{q-1}} \cdots [\mathrm{rad}(1, t)]^{b_1}$$

where

$$q = [\log_2(n)] + 1 \tag{6}$$

in which [·] means taking the greatest integer. Therefore,

$$n = b_q 2^{q-1} + b_{q-1} 2^{q-2} + \cdots + b_1 2^0$$

where $b_q b_{q-1} \cdots b_1$ is the binary expression of n.

Hence, if a particular Walsh function pal(n, t) is given and its Rademacher function components are required, we simply change n into binary form and then substitute into Eq. (6). For example, the Rademacher function components of Walsh function pal(10, t) is

$$\mathrm{pal}(10, t) = [\mathrm{rad}(4, t)]^1[\mathrm{rad}(3, t)]^0[\mathrm{rad}(2, t)]^1[\mathrm{rad}(1, t)]^0$$

where

$$q = [\log_2 10] + 1 = 4$$

because Rademacher functions are easy to draw, as are Walsh functions. Figure 4 shows the Walsh functions in Paley ordering from pal(0, t) to pal(7, t).

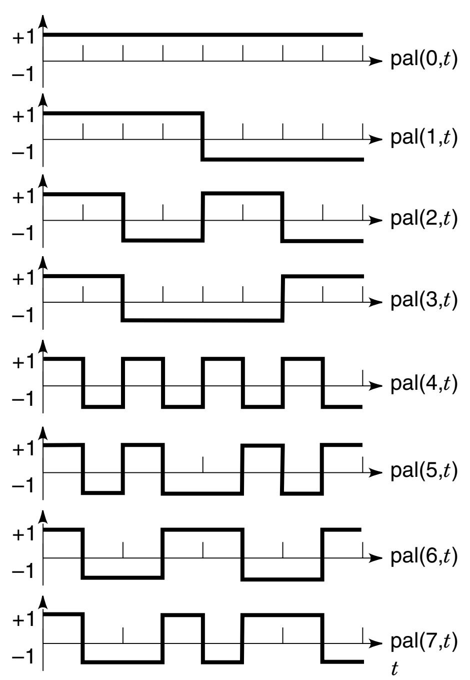

Figure 4. A set of the first eight Walsh functions arranged in dyadic order.

Since all Radmacher functions except rad(0, t) are odd functions about $t = 0.5$, they do not form a complete set. On the contrary, one can see that the Walsh functions constitute a complete orthonormal set of functions. The Walsh series representation of a function $f(t)$, which is absolutely integrable in [0, 1) in a dyadic ordering is

$$f(t) = \sum_{m=0}^{\infty} c_m \, \mathrm{pal}(m, t) \tag{7}$$

where

$$c_m = \int_0^1 f(t)\mathrm{pal}(m, t)\, dt \qquad m = 0, 1, \ldots \tag{8}$$

Let us now return to the Walsh coefficient evaluation in dyadic ordering for ramp function. Substituting $f(t) = t$ into Eqs. (7) and (8), we have

$$f(t) = t = \frac{1}{2}\mathrm{pal}(0,t) - \frac{1}{4}\mathrm{pal}(1,t) - \frac{1}{8}\mathrm{pal}(2,t) - \frac{1}{16}\mathrm{pal}(4,t) - \frac{1}{32}\mathrm{pal}(8,t) - \frac{1}{64}\mathrm{pal}(16,t) + \cdots \tag{9}$$

The original curve $f(t) = t$ and its Walsh series approximations are shown in Figure 5. They are stairways waves. The first representation is obtained by taking one term of the Walsh series, or $\frac{1}{2}$ pal(0, t); the second one is $\frac{1}{2}$ pal(0, t) $-$ $\frac{1}{4}$ pal(1, t). Figure 5 shows up to a four term approximation. From the coefficient evaluation process, we can easily see the similarities between the Fourier series and Walsh series.

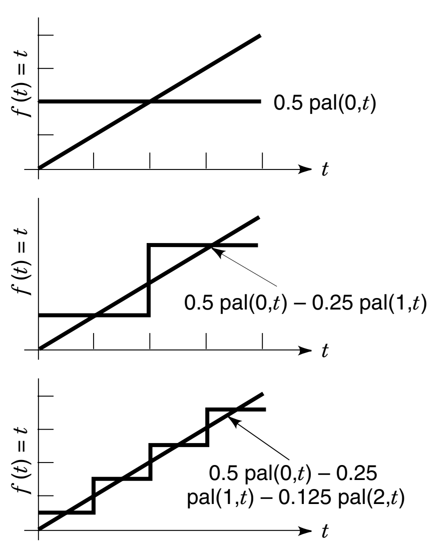

Figure 5. Expanding a ramp function into a Walsh series.

Natural or Hadamard Ordering

This ordering was originally proposed by Henderson (15) and follows the Hadamard matrix derived from successive Kronecker products. A Hadamard matrix is a square array whose coefficients comprise only +1 and −1 and in which the rows (and columns) are orthogonal to one another. In a symmetrical Hadamard matrix it is possible to interchange rows and columns or to change the sign of every element in a row without affecting these orthogonal properties. This makes it possible to obtain a symmetrical Hadamard matrix whose first row and first column contain only +1's. The matrix obtained in this way is known as the normal form for the Hadamard matrix. The lowest-order Hadamard matrix is of order two,

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Higher-order matrices, restricted to having powers of 2, can be obtained from the recursive relationship

$$H_N = H_{N/2} \otimes H_2$$

where $\otimes$ denotes the direct or Kronecker product (16) and N is a power of 2. In the Kronecker product each element in the matrix (in this case $H_{N/2}$) is replaced by the matrix H_2. Thus, for $N = 4$ we have

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

Furthermore, if we now replace each element in the H_4 matrix by an H_2 matrix we obtain an H_8 matrix. By replacing each row of this matrix by its equivalent naturally ordered Walsh functions we can form a series of functions which will indicate the ordering obtained through this derivation. Therefore, for a series consisting of eight terms we get

$$H_8 = H_4 \otimes H_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} \mathrm{had}(0,t) \\ \mathrm{had}(1,t) \\ \mathrm{had}(2,t) \\ \mathrm{had}(3,t) \\ \mathrm{had}(4,t) \\ \mathrm{had}(5,t) \\ \mathrm{had}(6,t) \\ \mathrm{had}(7,t) \end{bmatrix}$$

Relationship Between Ordered Series

The wal(i, t), pal(i, t), and had(i, t), $i = 0, 1, 2, \ldots$ ordered Walsh functions are related (1) through a bit reversal for the position of each component in a series, (2) through a conversion using a Gray code or (3) by a combination of both of these. For example, given a function numbered in dyadic ordering, the corresponding sequency order is given by

$$\mathrm{pal}(n, t) = \mathrm{wal}(b(n), t)$$

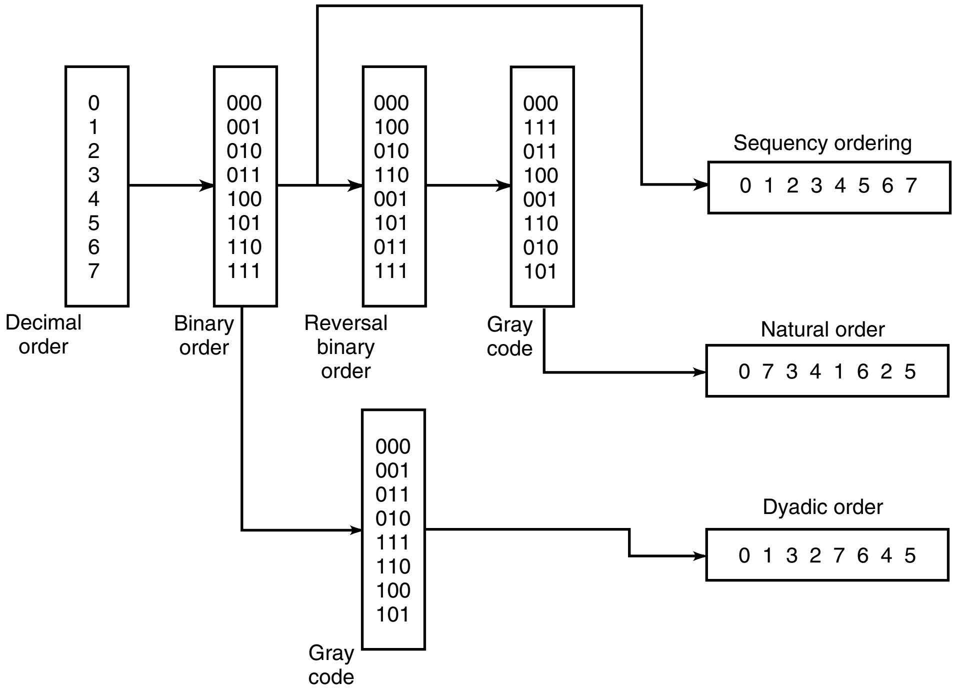

Figure 6. Relationships between three methods of ordering the Walsh functions series.

where $b(n)$ is a Gray-code-to-binary conversion of n. A procedure for carrying out this conversion is described by Yuen (17). These relationships for $N = 8$ are shown in Fig. 6 in which both dyadic and natural ordering are related to a sequency ordered.

Block-Pulse Functions

Block-pulse functions constitute another complete set of orthogonal basis functions. The type of approximation is the same as with Walsh functions, the only difference being in the simplicity of computations. The block-pulse function $b(i, t)$, $i = 1, 2, \ldots, m$ over a time interval $t \in [0, 1)$ is defined as

$$b(i,t) = \begin{cases} 1 & \dfrac{i-1}{m} \le t < \dfrac{i}{m} \quad i = 1, 2, \ldots, m \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

Thus, as shown in Fig. 7, the 8-order block-pulse functions are time functions having a unit height and $\frac{1}{8}$ width. By using the orthogonal property that is

$$\int_0^1 b(i,t)b(j,t)\,dt = \begin{cases} \dfrac{1}{m} & i = j \quad i, j = 1, 2, \ldots, m \\ 0 & i \ne j \end{cases}$$

A time function $f(t)$ which is absolutely integrable in $t \in [0, 1)$ can be approximately represented by a block-pulse series as

$$f(t) = \sum_{i=1}^{m} a_i b(i,t) = \gamma^T B(t)$$

where

$$\alpha_i = m \int_{i-1/m}^{i/m} f(t)\,dt \qquad i = 1, 2, \ldots, m$$

and

$$\gamma = [\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_m]^T$$
$$B(t) = [b(1,t) \quad b(2,t) \quad \cdots \quad b(m,t)]^T$$

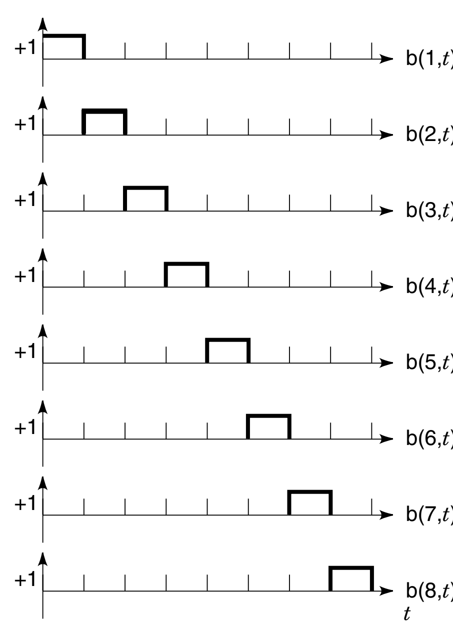

Figure 7. A set of the first eight block-pulse functions.

Relationship between Walsh and Block-Pulse Functions

A one-to-one relationship between Walsh and block-pulse functions was first offered in (18). In their work, Chen et al. used block-pulse operational matrices for simplifying their approach in Walsh domain analysis. The completeness of the block-pulse functions was first proved by Rao and Srinivasan (19) and later the convergent properties of the block-pulse series as well as its completeness were investigated by Kwong and Chen (20).

It was shown by Chen et al. (18) that the block-pulse functions $b(i, t)$ are related to the Walsh functions $\text{wal}(i, t)$ by the relation

$$\phi_b(t) = W_{(m\times m)}\phi_w(t)$$

where $W_{(m\times m)}$ is a square matrix of order m called the Walsh matrix, $\phi_b(t)$ and $\phi_w(t)$ are block-pulse and Walsh vectors respectively, defined by

$$\phi_b(t) = [b(1,t) \quad b(2,t) \quad \cdots \quad b(m,t)]^T$$
$$\phi_w(t) = [\text{wal}(0,t) \quad \text{wal}(1,t) \quad \cdots \quad \text{wal}(m-1,t)]^T$$

The Walsh matrix has the following property:

$$W^2_{(m\times m)} = mI_{(m)}$$

where $I_{(m)}$ is a unit matrix of order m. For $m = 8$ Walsh matrix is given by Eq. (1).

Construction of any function in time domain is easier with block-pulse functions rather than with Walsh functions. If a function is represented by block-pulse functions, then the amplitude of each block-pulse in any sub-interval represents the average value of that function in that particular time interval.

Some properties of the block-pulse functions are (1) they form a complete orthogonal set which could easily be normalized, (2) they are computationally much simpler, and yet produce the same numerical accuracy as that obtained by the Walsh function approach, (3) any number of block-pulse functions can be used to form a complete set, while Walsh functions require 2^p ($p = 1, 2, \ldots$) component functions to form a set suitable for analytical manipulations, and (4) they need less computation time as well as computer memory space than Walsh functions (21–23).

WALSH FUNCTION GENERATOR

The Walsh function and Walsh transform are important analytical and hardware tools for signal processing. They have found wide application in digital communications (24) as well as in system analysis (25). Walsh function generators have frequent use in many areas of electrical engineering. Implementation of such generators through hardware logic gives rise to orthogonality error. Orthogonality error is the shift of the transition points of the Walsh functions of Fig. 3 or Fig. 4 from their assigned places in the time scale.

The sequency generators having the widest applicability are those generating a set of Walsh series, although in some cases the series are obtained by first generating a series of Radmacher functions. A generator that produces a set of m Walsh functions $\text{wal}(n, t)$ where $n = 0, 1, \ldots, m - 1$, is called an array generator. Ideally the generated waves will be orthogonal to each other, and some designs are better in achieving this than are others.

Two classifications of array generators are considered. The first generates fixed sets of Walsh functions $\text{wal}(n, t)$, where only the sequency range of the entire array is controlled externally. These array generators find use in multiplexing and signal processing. The second classification includes generators for which the sequency order n and/or the time interval t are controlled externally. These are known as programmable generators. Further subclassifications of programmable generators can be defined, namely, serial programmable generators in which the time interval is fixed and the sequency order controlled and parallel programmable generators in which the sequency range is fixed and the time interval controlled.

A global Walsh generator capable of producing three different ordered outputs. These outputs are natural, sequency, and dyadic (26). This generator generates Walsh functions through logical combinations of Rademacher functions. However, these methods are all implemented using hardware digital logic and sequential circuits.

The use of microprocessors for the generation of global Walsh functions provides wider flexibility for low-cost applications which can be controlled by supporting software with better accuracy and much wider versatility. In system analysis, where the Walsh function technique provides easier mathematical manipulations, for example, power-electronic systems (27), this kind of generator can be used to study system behavior where slow speed of software based generation does not hinder the time responses.

RELATIONSHIP BETWEEN WALSH AND FOURIER SERIES

When the Walsh series representation of a time signal is required to be converted to the more familiar Fourier series representations, then the Fourier transforms of the Walsh functions are needed in the conversion equations. A nonrecursive algorithm by Siemens and Kitai (28) is used to set up the necessary conversion coefficients. A recursive formula by Blachman (29) is used to evaluate the Walsh transforms of sinusoids. This formula can also be modified to yield the Fourier transform of Walsh functions. The result in both cases is the same, although in the former case the resulting conversion matrix is more easily obtainable. A brief review of Fourier series follows.

A periodic function $f(t)$ defined over the interval 0 to 1 may be expanded into Fourier series as follows

$$f(t) = a_0 + \sum_{n=1}^{\infty}\{a_n\cos(2n\pi t) + a^*_n\sin(2n\pi t)\} \qquad (11)$$

where the Fourier coefficients a_n and a^*_n are given by

$$a_0 = \int_0^1 f(t)\,dt$$
$$a_n = 2\int_0^1 f(t)\cos(2n\pi t)\,dt \qquad n = 1, 2, \ldots$$
$$a^*_n = 2\int_0^1 f(t)\sin(2n\pi t)\,dt \qquad n = 1, 2, \ldots$$

if $f(t)$ is truncated up to its first $2r + 1$ terms, then Eq. (11) can be written as

$$f(t) = a_0 + \sum_{n=1}^{r}\{a_n \cos(2n\pi t) + a_n^* \sin(2n\pi t)\} = A^T\Psi(t) \quad (12)$$

where the Fourier series coefficient vector A and the Fourier series vector $\Psi(t)$ are defined as

$$A = [a_0 \quad a_1 \quad a_2 \quad \cdots \quad a_r \quad a_1^* \quad a_2^* \quad \cdots \quad a_r^*]^T$$
$$\Psi(t) = [\phi_0(t) \quad \phi_1(t) \quad \cdots \quad \phi_r(t) \quad \phi_1^*(t) \quad \cdots \quad \phi_r^*(t)]^T$$

with

$$\phi_n(t) = \cos(2n\pi t) \qquad n = 0, 1, 2, \ldots$$
$$\phi_n^*(t) = \sin(2n\pi t) \qquad n = 1, 2, \ldots$$

The elements of $\Psi(t)$ are orthogonal in the interval $t \in [0, 1)$.

The sal and cal terms defined in Eqs. (2) and (3) for the Walsh functions are analogous to sine and cosine terms in the Fourier series, respectively. In a similar fashion to Fourier series expansion by truncating Eq. (4), any time signal $f(t)$ can be expressed as a sum of Walsh functions as

$$f(t) = d_0\text{wal}(0,t) + \sum_{i=1}^{m-1} d_i\text{wal}(i,t) = D^T\phi_w(t) \qquad (13)$$

where

$$d_0 = 2\int_0^1 f(t)\text{wal}(0,t)\,dt$$
$$d_1 = \int_0^1 f(t)\text{wal}(i,t)\,dt, \quad i = 1, \ldots, m-1$$

and

$$D = [d_0 \quad d_1 \quad \cdots \quad d_{m-1}]^T$$

Using Eqs. (11) and (12), the following expression holds

$$B\phi_w(t) = \Psi(t)$$

where B is the Fourier–Walsh conversion matrix.

The inverse relation is also valid

$$\phi_w(t) = B^{-1}\Psi(t)$$

The following steps can be used to create the conversion matrix B:

1. For Walsh function order number v, obtain its binary equivalent expression b.
2. Convert b to its Gray code equivalent y.
3. The total number of bits in y is h and the number of bits with the binary value 1 in y is d.
4. The Fourier coefficient of order u of the Walsh function of order v appears as the element in row u and column v of the conversion matrix B. These elements can be calculated according to the following equations (30).

$$B(u,v) = 2(-1)^{y_0}(-j)^d\left[\prod_{w=0}^{h-1}\cos\left(\frac{u\pi}{2^{u+1}} - \frac{y_w\pi}{2}\right)\right] \times \frac{\sin\left(\frac{u\pi}{2^h}\right)}{\frac{u\pi}{2^h}}$$

where y_0 is the least significant bit in the Gray code expression of y and $j = \sqrt{-1}$.

The sequency-ordered matrix B of order 8×8 can be obtained as follows (30,31):

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.27 & 0 & -0.527 & 0 & -0.105 & 0 & -0.253 \\ 0 & 0 & 1.27 & 0 & 0 & 0 & -0.527 & 0 \\ 0 & 0.424 & 0 & 1.02 & 0 & -0.685 & 0 & 0.284 \\ 0 & 0 & 0 & 0 & 1.27 & 0 & 0 & 0 \\ 0 & 0.255 & 0 & 0.615 & 0 & 0.092 & 0 & -0.381 \\ 0 & 0 & 0.424 & 0 & 0 & 0 & 1.02 & 0 \\ 0 & 0.182 & 0 & -0.007 & 0 & 0.379 & 0 & 0.914 \end{bmatrix}$$

Relationship between Block-Pulse and Fourier Series

Using Eq. (10), the Fourier series for $b(n, t)$ is given by

$$b(n,t) = b_0 + \sum_{i=1}^{k}[b_{ni}\cos(2i\pi t) + b_{ni}^*\sin(2i\pi t)] \qquad (14)$$

where

$$b_0 = \int_0^1 b(n,t)\,dt = \frac{1}{m} \qquad (15)$$

$$\begin{aligned} b_{ni} &= 2\int_0^1 b(n,t)\cos(2\pi it)\,dt \\ &= \frac{2}{\pi i}\sin\left(\frac{\pi i}{m}\right)\cos\left(\frac{\pi i}{m}(2n-1)\right) \qquad \begin{matrix} i = 1, \ldots, k \\ n = 1, \ldots, m \end{matrix} \end{aligned} \qquad (16)$$

and

$$\begin{aligned} b_{ni}^* &= 2\int_0^1 b(n,t)\sin(2\pi it)\,dt \\ &= \frac{2}{\pi i}\sin\left(\frac{\pi i}{m}\right)\sin\left(\frac{\pi i}{m}(2n-1)\right) \qquad \begin{matrix} i = 1, \ldots, k \\ n = 1, \ldots, m \end{matrix} \end{aligned} \qquad (17)$$

Using Eqs. (14–17) the following expression holds

$$B(t) = R\Psi(t)$$

where R is the $m \times (1 + 2k)$ Fourier block-pulse conversion matrix given by

$$R = \begin{bmatrix} b_0 & b_{11} & \cdots & b_{ik} & b_{11}^* & \cdots & b_{1k}^* \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_0 & b_{m1} & \cdots & b_{mk} & b_{m1}^* & \cdots & b_{mk}^* \end{bmatrix}$$

and

$$\Psi(t) = [1 \quad \cos(2\pi t) \quad \cdots \quad \cos(2k\pi t) \quad \sin(2\pi t) \quad \cdots \quad \sin(2k\pi t)]^T$$

For example for $k = 5$ and $m = 6$, we get

$$R = \frac{1}{6}\begin{bmatrix} 1 & 1.654 & 0.827 & 0 & -0.414 & -0.331 & 0.955 & 1.432 & 1.273 & 0.716 & 0.191 \\ 1 & 0 & -1.654 & 0 & 0.827 & 0 & 1.910 & 0 & -1.273 & 0 & 0.382 \\ 1 & -1.654 & 0.827 & 0 & -0.414 & 0.331 & 0.955 & -1.432 & 1.273 & -0.716 & 0.191 \\ 1 & -1.654 & 0.827 & 0 & -0.414 & 0.331 & -0.955 & 1.432 & -1.273 & 0.716 & -0.191 \\ 1 & 0 & -1.654 & 0 & 0.827 & 0 & -1.910 & 0 & 1.273 & 0 & -0.382 \\ 1 & 1.654 & 0.827 & 0 & -0.414 & -0.331 & -0.955 & -1.432 & -1.273 & -0.716 & -0.191 \end{bmatrix}$$

using Eq. (12), we get

$$A = R^T C$$

Application of Walsh Functions in Dynamic Systems, Identification, and Control

The initiation of the analysis of dynamic systems in the time domain via Walsh functions was made by Corrington in 1973 (32) in his paper on the solution of differential and integral equations. The key idea was the observation that successive integrals of Walsh functions are expressed as Walsh series with well-defined, tabulated coefficients. The differential equation is solved for the highest derivative, and the result is then integrated as many times as required to give the solution. Two years later, Chen and Hsiao (33) presented the solution of dynamic systems in state space formulation by a more systematic use of the Walsh function integration property expressed by an operational equation as

$$\int_0^t P(t)\,dt = EP(t)$$

where

$$P(t) = [\mathrm{pal}(0,t) \quad \mathrm{pal}(1,t) \quad \cdots \quad \mathrm{pal}(n-1,t)]^T$$

and E is a well-defined operational matrix. Using this operational equation, the state-space differential system is converted to a linear algebraic system, which has to be solved for a set of unknown Walsh series coefficients. In what follows the operational matrix for Walsh functions will be briefly discussed.

Let us take pal(0, t), pal(1, t), . . ., pal(7, t) and integrate them; we will have various triangular waves (33). If we evaluate the Walsh coefficient for these triangular waves, we get the following matrix for $E_{(8\times 8)}$:

$$E_{(8\times 8)} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{8} & 0 & -\frac{1}{16} & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & -\frac{1}{8} & 0 & -\frac{1}{16} & 0 & 0 \\ \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{16} & 0 \\ 0 & \frac{1}{8} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{16} \\ \frac{1}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{16} & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is preferable to make the dimension of the matrix equal to 2^n, where n is an integer. Making this choice will enable us to obtain simpler results. It is noted that

$$\int_0^t \mathrm{pal}(0,t)\,dt = t$$

therefore, the first row of E is the first four terms of Eq. (9). A general formula for $E_{(n\times n)}$ can be written as

$$E_{(n\times n)} = \left[\begin{array}{cc:c:c} \frac{1}{2} & -\frac{2}{n}I_{n/8} & -\frac{1}{n}I_{n/4} & -\frac{1}{2n}I_{n/2} \\ \frac{2}{n}I_{n/8} & 0_{n/8} & & \\ \hdashline \multicolumn{2}{c:}{\frac{1}{n}I_{n/4}} & 0_{n/4} & \\ \hdashline \multicolumn{3}{c:}{\frac{1}{2n}I_{n/2}} & 0_{n/2} \end{array}\right] \qquad (18)$$

It is interesting to note that if Eq. (18) is partitioned into four parts as shown, the upper left part of $E_{(n\times n)}$ is identical to

$$E_{(\frac{n}{2}\times\frac{n}{2})}$$

and the upper left corner of

$$E_{(\frac{n}{2}\times\frac{n}{2})} \quad \text{is} \quad E_{(\frac{n}{4}\times\frac{n}{4})}$$

Therefore, this regularity of the structure of the E matrix enables us to write the nth enlarged matrix to any dimension, if the dimension number is restricted to 2^n where n is an integer number.

Let us illustrate the application of operational matrix of integration by solving the following state equation

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(0) = x_0$$

where $x(t)$ is a state vector of n components and $u(t)$ is an input vector of l components. A and B are $n \times n$ and $n \times l$ matrices, respectively. We now solve the state equations via Walsh series.

First we assume the rate variable vector $\dot{x}(t)$ can be expressed as

$$\dot{x}(t) = [c_0 \quad c_1 \quad \cdots \quad c_{m-1}]P(t) = CP(t)$$

where each c_i, $i = 1, \ldots, m - 1$, is an n vector. The state variable $x(t)$ may be obtained as

$$x(t) = C\int_0^t P(t)\,dt + x_0$$

Also the input vector can be expressed by Walsh series as

$$u(t) = HP(t)$$

where H is a $l \times m$ matrix. Thus we get

$$CP(t) = A(CEP(t) + x_0) + BHP(t)$$

Ax_0 can also be written as

$$Ax_0 = Ax_0P(t) = [Ax_0 \quad 0 \quad \cdots \quad 0]P(t) = GP(t)$$

Finally we have

$$C = ACE + G + BH$$

hence

$$C = ACE + K$$

where

$$G + BH = K$$

If we arrange the $n \times m$ matrix C as an nm vector c by changing its first column into the first n components of the vector; the second column, the second n components of the vector, and so on; and rearrange K in the same manner; we obtain

$$c = (A \otimes E^T)c + k \tag{19}$$

where $\otimes$ denote Kronecker product. Using Eq. (19), the solution of c is

$$c = [I - A \otimes E^T]^{-1}k$$

once c has been decided, the Walsh series representation for the rate variable is determined. The state variable vector is then found by substitution.

In addition to being applied to system analysis Walsh function expansions have also been applied with success to the design and implementation of optimal filter and controllers, naturally providing piecewise constant approximations of the optimal feedback gains. Previously, such approximations were determined by prespecifying the structural form of the time varying gains by Kleinman, Fortmann, and Athans in 1968 (34). The idea in using Walsh function series in optimal control problems was first employed by Chen and Hsiao (33). Essentially, the method belongs to the direct variational approach and is very powerful and easily implemenable. Tzafestas and Stavroulakis (35) used finite Walsh series expansion for designing approximate (suboptimal) observers and filters incorporated in a close-loop optimal controller. Both continuous and discrete time systems, time-invariant, and time-varying are considered. The solution provides a computational algorithm that gives the Walsh expansion coefficients of the state and observer output. Further, Chen and Hsiao applied Walsh functions (1) to solve the problems of linear systems by the state space model (36), (2) for time domain synthesis (37), (3) To solve the optimal control problem (38), (4) in the variational problem (39), and (5) for fractional calculus as applied to distributed systems (40).

Additionally, Walsh functions proved to be very powerful in solving the identification (or synthesis) problem of dynamic systems from given input-output records. The paper by Chen and Hsiao (37) appears to be the first work in which the problem of identifying dynamic systems is solved with the aid of Walsh functions. The key idea is the application of repeated integration together with the Walsh operational matrix employed for determining the system response. In Ref. 41, bilinear system identification is considered and solved by using the Walsh operational matrix and the group properties of Walsh functions. The same type of systems were also researched by Chen and Shih (42).

Rao and Palanisamy (43) provides a methodology for improving the identification accuracy of continuous systems through the so-called one-shot operational matrices for repeated integration via Walsh functions. Further, a multistep parameter estimation algorithm is given in Ref. 43 for systems with large, unknown time delays. Some additional works in the area of systems identification via Walsh functions are described in Rao and Sivakumar (44), Gopalsami and Deekshatulu (45), Tzafestas and Chrysochoides (46), and Tzafestas, Papastergoius, and Anoussis (47).

Moreover, Rao used Walsh function for (1) optimal control of time delay systems (48) (2) identification of time-lag systems (49) (3) transfer function matrix identification (50) (4) parameter estimation (25) (5) solving functional differential equations and related problems (51). Rao and Tzafestas (52) indicated the potentiality of Walsh and related functions in the area of systems and control in a review paper.

W. L. Chen defined a shift Walsh matrix for solving delay-differential equations (53) and used Walsh functions for parameter estimation of bilinear systems (42) as well as in the analysis of multidelay systems (54). Paraskevopoulos determined the transfer function of a single input single output (SISO) system from its impulse response with the help of Walsh functions and a fast Walsh algorithm (55). Tzafestas applied a Walsh series approach for lumped and distributed system identification (56). Mahapatra used Walsh functions for solving matrix Riccati equation arising in optimal control studies of linear diffusion equation (57). Mouldeens work was concerned with the application of Walsh spectral analysis of ordinary differential equations in a very formal as well as mathematical manner (58). Deb and Datta was the first to define Walsh operational transfer function for analysis of linear SISO systems (27,59) and Deb, Sen, and Datta (60) gave a review paper in Walsh functions and their applications in 1992.

The mathematical basis of Walsh function methods has become strong and versatile enough to encourage their application to the analysis of power-electronic circuits, and systems (31–61). From the study of different aspects of the Walsh functions, we find the following properties suitable for application to the analysis power-electronic systems

1. Any member of the Walsh-function set resembles, to some extent, the typical switching pattern of a power-electronic converter. Hence, the voltage output of such a converter can be well represented by Walsh functions.
2. Walsh functions are defined in time domain. Thus, we do not need any inverse transformation as we do in Laplace domain analysis.
3. The set of Walsh functions is complete and orthonormal, thereby offering the facility for easier mathematical manipulations, including the design of fast computational algorithm.

Application of Walsh Functions in Different Areas of Science and Technology

Scientists have found that the binary nature of Walsh functions and its striking similarity to the familiar sine–cosine functions could adapt it for application in many areas of science and technology.

In the early 1960s, the first significant application of a Walsh function in the field of communications was noted. The credit goes mostly to Harmuth and his associates (62–64). Consequently, the Walsh functions were found to be an efficient tool in the field of signal multiplexing. Several experimental multiplexing systems were developed which made use of this nonsinusoidal technology.

In the multiplexing scheme, several independent signals are sent via a common communication channel. Walsh functions as carriers of communication signals are used in multiplexing schemes. While any set of complete orthogonal functions can be used as carriers, practical difficulties restrict the use of orthogonal functions to Walsh, block-pulse, and sinusoidal functions only. In time division multiplexing (TDM) block-pulses are generally used as carriers. Frequency division multiplexing (FDM) generally uses sinusoidal functions as carriers. However, in sequency division multiplexing (SDM) schemes Walsh functions are used as carriers.

One of the major advantages of using Walsh functions as carriers is that the multiplication process produces only one side-band and not two, as in the case with sinusoidal products. The reason is that the Walsh functions form a group under multiplication, that is, the product of two Walsh functions results in another Walsh function. The application of sequency digital processing technique covers a very wide field and includes various uses of spectral analysis. Some of these areas are, radar and sonar application, medical signal processing, speech processing, digital image processing and optics communications (65–68).

The processing of fixed or changing visual images by using digital techniques requires the manipulation of multidimensional signals involving operations on large numbers of data values. Since this process generally involves high speed online computer and substantial processing time, serious attempts have been made to find efficient alternative techniques. The sequency techniques (especially two-dimensional Walsh functions, with their emphasis on rapid computational ease) play significant roles in these development.

Considerable development in applying Walsh functions has been carried out by Hurst (69) and others during the 1970s. The techniques of domain analysis have also led to their use in the design of higher logic functions such as threshold logic gates (70). Here a subset of Walsh series, known as the Chow parameters (71), have proved particularly useful. In addition to these, application of Walsh functions has expanded to the formulation of multiinput gate structures (72), digital system fault diagnosis (73), digital circuit synthesis (74), and other related areas.

The widespread interest in practical applications of Walsh functions has stimulated further contributions to the mathematical theory. Of special interest is the logical differential calculus of Gibbs (75–76). In contrast to the sine-cosine functions, which often represent the characteristic solution to certain linear differential equations, the Walsh functions are shown to represent the solutions to what is known as the logical differential equations. Applications of Gibbs derivative are found in mathematical logic (77), approximation theory (78), statistics (79–80), and linear system theory (81).

SUMMARY

With the efforts of many researchers during the past twenty five years, the mathematical basis of Walsh functions methods has become strong and versatile. This basis encourages the application of these methods to the analysis of problems related to circuits, systems, and communications. When the analysis is carried out in the sequency domain instead of frequency domain, the method is straightforward, elegant, and compatible with easy computer manipulations.

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