Z TRANSFORMS

One of the most useful techniques in engineering analysis is transforming a problem from the time domain to the frequency domain. Using a Fourier transform, differential equations are changed to difference equations, often substantially simplifying the analysis. When dealing with linear systems, the relationship between the input and output is a convolution integral. However, this reduces to simple multiplication in the frequency domain. When dealing with transient signals, it is often more convenient to use the Laplace transform, but the principle is the same.

The Laplace and Fourier transforms are appropriate for analog signals. When dealing with digital signals, the Z transform is used. The reasons for using it are analogous: (1) complicated difference equations in the time domain become algebraic equations in the Z domain, and (2) the relationship between the input and output of a linear system is a multiplication in the Z domain instead of a convolution in the sampled time domain.

DEFINITION OF THE *Z* TRANSFORM

The Z transform is extremely useful when dealing with functions in the sampled time domain; that is, instead of the function $x(t)$, we have a function of the type

$$x(n) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT)$$

where T is a uniform time interval. The Z transform is defined by

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{1}$$

A power of z is associated with each delay interval T. Suppose we have a function $x(n)$, as given in Fig. 1(a)

$$x(n) = \delta(t) + 0.5\delta(t - T) + 0.25\delta(t - 2T) \tag{2}$$

In the Z domain, this is written as

$$X(z) = 1 + 0.5z^{-1} + 0.25z^{-2} \tag{3}$$

which is shown in Fig. 1(b). Note that the first term is merely 1, because $z^{-0} = 1$.

In its simplest form, the z^{-1} operator associated with the Z transform can be thought of as a delay operator. So if

$$y(n) = x(n - 1)$$

then the Z transform of this equation would be written as

$$Y(z) = z^{-1}X(z)$$

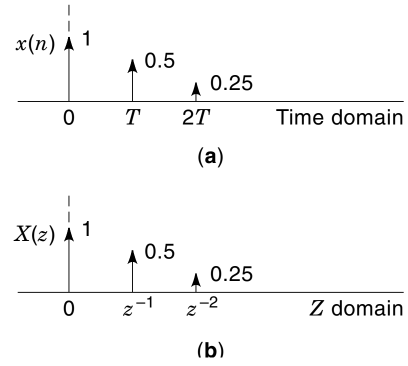

Figure 1. Graph of Eq. (2) in (a) the sampled time domain and (b) the Z domain. Notice that the Z domain is exactly the same as the sampled time domain, except that delays by the time interval T are indicated by the z^{-1} operator.

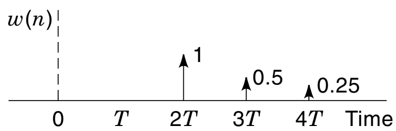

Figure 2. Graph of Eq. (4) after it has been put in the time domain. Since $W(z) = z^{-2}X(z)$, the graph of $w(n)$ is the same as $x(n)$ in Fig. 1, except it has been delayed by $2T$, as indicated by the z^{-2}.

and if

$$w(n) = y(n-1)$$

the Z transform of $w(n)$ is

$$W(z) = z^{-1}Y(z) = z^{-1}\left(z^{-1}X(z)\right) = z^{-2}X(z) \tag{4}$$

If $W(z)$ is defined as in Eq. (3), and $X(z)$ as defined in Eq. (2) then

$$W(z) = z^{-2}1 + 0.5z^{-3} + 0.25z^{-4}$$

which in turn, going back to the sampled time domain (Fig. 2), means that

$$w(n) = \delta(t-2T) + 0.5\cdot\delta(t-3T) + 0.25\cdot\delta(t-4T)$$

We can add two Z transforms, merely by making sure like powers of z are lumped together. For instance, adding $X(z)$ of Eq. (2) and $W(z)$ of Eq. (3) gives

$$\begin{aligned} X(z)+W(z) &= [1+0.5z^{-1}+0.25z^{-2}]+[1z^{-2}+0.5z^{-3}+0.25z^{-4}] \\ &= 1+0.5z^{-1}+1.25z^{-2}+0.5z^{-3}+0.25z^{-4} \end{aligned}$$

Going back to the sampled time domain,

$$\begin{aligned} x(n)+w(n) =& \,\delta(t) + 0.5\cdot\delta(t-T) + 1.25\cdot\delta(t-2T) \\ &+ 0.5\cdot\delta(t-3T) + 0.25\cdot\delta(t-4T) \end{aligned}$$

which could have been obtained by adding $x(n)$ and $w(n)$, being sure to keep like delta (δ) terms together.

CONVOLUTION USING THE *Z* TRANSFORM

Figure 3 illustrates a simple linear system. In this example, suppose $h(n)$ is a system that adds the present value of $x(n)$ to its previous value $x(n-1)$ and outputs it as the new value of $y(n)$:

$$y(n) = x(n) + x(n-1) \tag{5}$$

This function $h(n)$ is referred to as the "impulse response" for the following reason: if the input $x(n)$ is an impulse $\delta(n)$ then

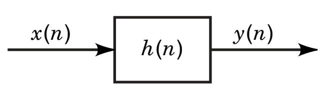

Figure 3. A simple discrete linear system.

the output is

$$\begin{aligned} y(n) &= 1 \qquad \text{for } n = 0 \text{ or } 1 \\ y(n) &= 0 \qquad \text{otherwise} \end{aligned}$$

See Ref. 1 or 2. Therefore,

$$h(n) = \delta(n) + \delta(n-1) \tag{6}$$

A more mathematically concise expression is

$$y(n) = \sum_{i=0}^{1} x(n-i)\cdot h(i) \tag{7}$$

Equation 7 is a discrete convolution. Notice that the index i only ranges between 0 and 1 because in Eq. (6) $h(n)$ has only these two terms. This could be generalized for any upper limit to infinity.

If, instead of an impulse, we use the values of $x(n)$ from Eq. (2), $y(n)$ is calculated from Eq. (7):

$$y(n) = \delta(t) + 1.5\cdot\delta(t-T) + 0.75\cdot\delta(t-2T) + 0.25\cdot\delta(t-3T) \tag{8}$$

This process was made tractable only by the small number of terms used in this example.

As an alternative approach, take the Z transforms of $x(n)$ and $h(n)$:

$$\begin{aligned} X(z) &= 1 + 0.5\cdot z^{-1} + 0.25\cdot z^{-2} \\ H(z) &= 1 + z^{-1} \end{aligned}$$

Multiplying the two together gives

$$\begin{aligned} H(z)\cdot X(z) &= (1+0.5\cdot z^{-1}+0.25\cdot z^{-2})\cdot(1+z^{-1}) \\ &= 1+1.5\cdot z^{-1}+0.75\cdot z^{-2}+0.25\cdot z^{-2} \end{aligned}$$

Going back to the time domain gives the $y(n)$ of Eq. (8). This illustrates the powerful "convolution theorem": Convolution in the discrete time domain becomes multiplication in the Z domain. The $H(z)$, which is the Z transform of the impulse response, is referred to as the transfer function. The proof follows.

Starting with the definition of convolution in the discrete time domain,

$$y(n) = \sum_{i=0}^{\infty} h(n-i)\cdot x(i) \tag{9}$$

take the Z transform of both sides

$$\sum_{n=0}^{\infty} y(n)z^{-n} = \sum_{n=0}^{\infty}\sum_{i=0}^{\infty} h(n-i)\cdot x(i)z^{-n}$$

and then interchange the summation signs

$$Y(z) = \sum_{i=0}^{\infty} x(i)\sum_{n=0}^{\infty} h(n-i)\cdot z^{-n}$$

Finally, multiplying by $z^{-i}\cdot z^{i}$ gives

$$Y(z) = \sum_{i=0}^{\infty} x(i)z^{-1}\sum_{n=0}^{\infty} h(n-i)\cdot z^{-n+i}$$

and using the parameter $m = n - i$

$$Y(z) = \sum_{i=0}^{\infty} x(i) \cdot z^{-i} \sum_{m=0}^{\infty} h(m) \cdot z^{-m}$$

gives

$$Y(z) = H(z) \cdot Y(z) \tag{10}$$

Note that Eq. (1) is usually referred to as the *bilateral Z Transform* because it is defined for both positive and negative n. However, we will almost always use causal functions, so the summation will be over the positive n's.

Example

As an example, suppose the impulse response in Fig. 3 is an exponentially decaying function

$$h(n) = Ae^{-nT/t_0} \qquad n = 0, 1, 2, 3, \ldots \tag{11}$$

and the input is the discretized unit step function

$$u(n) = 1 \qquad n = 0, 1, 2, 3, \ldots \tag{12}$$

Since

$$\sum_{n=0}^{\infty} a^{-n} = \frac{1}{1 - a^{-1}} \quad \text{when} \quad a \le 1$$

$H(z)$ can be calculated

$$Z[h(n)] = H(z) = A \sum_{n=0}^{\infty} [e^{T/t_0} z]^{-n} = \frac{A}{1 - e^{-T/t_0} z^{-1}} \tag{13}$$

and similarly

$$Z[u(n)] = U(z) = \frac{1}{1 - z^{-1}}$$

Or we could simply refer to a table of Z transforms, such as Table 1. The desired convolution is

$$\begin{aligned} Y(z) = H(z) \cdot U(z) &= \frac{A}{1 - e^{-T/t_0} z^{-1}} \cdot \frac{1}{1 - z^{-1}} \\ &= \frac{A}{1 - (1 + e^{-T/t_0}) z^{-1} + e^{-T/t_0} z^{-2}} \end{aligned} \tag{14}$$

To get a solution in the time domain, we take the partial fraction expansion of $Y(z)$

$$Y(z) = \frac{A}{1 - e^{-T/t_0} z^{-1}} \cdot \frac{1}{1 - z^{-1}} = A \cdot \left[\frac{B}{1 - e^{-T/t_0} z^{-1}} + \frac{C}{1 - z^{-1}} \right] \tag{15}$$

It turns out that

$$B = -\frac{e^{-T/t_0}}{1 - e^{-T/t_0}} \quad \text{and} \quad C = \frac{1}{1 - e^{-T/t_0}}$$

(The partial fraction expansion technique will be explained later.) Now the two terms in Eq. (15) can be taken back into the sampled time domain by finding the time domain terms corresponding to the two Z domain terms in Table 1, giving

$$y(n) = \frac{A}{1 - e^{-T/t_0}} [1 - e^{-(n+1)T/t_0}] \quad n = 0, 1, 2, 3 \ldots \tag{16}$$

Equation (16) is an analytic solution. An alternative approach exists. Consider Eq. (14) as a purely algebraic problem where we are solving for $Y(z)$:

$$\begin{aligned} [1 - (1 + e^{-T/t_0}) z^{-1} - e^{-T/t_0} z^{-2}] \cdot Y(z) &= A \\ Y(z) = (1 + e^{-T/t_0}) z^{-1} Y(z) - e^{-T/t_0} z^{-2} Y(z) &+ A \end{aligned} \tag{17}$$

Then remembering that the z^{-1} is an operator that just means a delay of one, we can go back to the sampled time domain

$$y(n) = (1 + e^{-T/t_0}) y(n-1) - e^{-T/t_0} y(n-2) + A \cdot \delta(n) \tag{18}$$

Note the following: The A term of Eq. (17) became $A \cdot \delta(n)$ because a constant in the Z domain is a delta function in the

Table 1. Transforms Among the Time, Frequency, Sampled-Time, and Z Domains

Time Domain	Frequency Domain	Sampled Time Domain	Z Domain
$\delta(t)$	1	$\delta(n)$	1
$u(t)$	$\frac{1}{j\omega}$	$u(n)$	$\frac{1}{1 - z^{-1}}$
$tu(t)$	$\frac{1}{(j\omega)^2}$	$n \cdot u(n)$	$\frac{z^{-1}}{(1 - z^{-1})^2}$
$e^{-\alpha t} \cdot u(t)$	$\frac{1}{\alpha + j\omega}$	$e^{-\alpha n T} \cdot u(n)$	$\frac{1}{1 - z^{-1} e^{-\alpha \cdot T}}$
$e^{-\alpha t} \sin(\beta t) \cdot u(t)$	$\frac{\beta}{(\alpha^2 + \beta^2) + j2\alpha\omega - \omega^2}$	$e^{-\alpha n T} \sin(\beta n T) \cdot u(n)$	$\frac{e^{-\alpha \cdot T} \cdot \sin(\beta T) \cdot z^{-1}}{1 - 2e^{-\alpha \cdot T} \cdot \cos(\beta T) \cdot z^{-1} + e^{-2\alpha \cdot T} \cdot z^{-2}}$
$e^{-\alpha t} \cos(\beta t) \cdot u(t)$	$\frac{\alpha + j\omega}{(\alpha^2 + \beta^2) + j2\alpha\omega - \omega^2}$	$e^{-\alpha n T} \cos(\beta n T) \cdot u(n)$	$\frac{1 - e^{-\alpha \cdot T} \cdot \cos(\beta T) \cdot z^{-1}}{1 - 2e^{-\alpha \cdot T} \cdot \cos(\beta T) \cdot z^{-1} + e^{-2\alpha \cdot T} \cdot z^{-2}}$

```
DELTA = 1
DO N=0,NMAX
  Y(N) = (1 + EXP(−T/T0) ) * Y(N−1) − EXP(−T/T0)*Y(N−2) + A*DELTA
  DELTA = 0
END DO
```

(a)

```
X = 1
DO N=0,NMAX
    Y(N) = EXP(−T/T0) * Y(N−1) + A* X
END DO
```

(b)

Figure 4. Computer codes to convolve an exponentially decaying function and the unit step function: (a) implementation of Eq. (18); (b) implementation of Eq. (19). The two codes are apparently of different form, but give identical results.

time domain (Table 1). To convince yourself of this, substitute $\delta(n)$ for $x(n)$ in Eq. (1), the definition of the Z transform. Only the $n = 0$ term survives, that is, a constant. It is not obvious, but Eq. (18) is equivalent to Eq. (16). Equation (16) is an analytic solution, while Eq. (18) is more appropriate when calculating the solution iteratively. A computer code to calculate Eq. (17) is given in Fig. 4(a).

There is yet another alternative approach. In the previous example, we specified the input $x(n)$ in Fig. 1, as $u(n)$. This is the step function, sometimes referred to as the Heaviside function. Suppose $x(n)$ is left as an unspecified function. Then

$$Y(z) = H(z)X(z) = \frac{A}{1 - e^{-T/t_0}z^{-1}} \cdot X(z)$$

and following the same process

$$\begin{aligned} Y(z)(1 - e^{-T/t_0}z^{-1}) &= A \cdot X(z) \\ Y(z) &= e^{-T/t_0}z^{-1}Y(z) + A \cdot X(z) \\ y(n) &= e^{-T/t_0}y(n-1) + A \cdot x(n) \end{aligned} \tag{19}$$

Now the appropriate computer code is given by Fig. 4(b). The result is identical to that generated in Fig. 4(a); however, this is the more general form. The function x is just specified as 1, and assuming that $N = 0$ corresponds to $t = 0$, X is the step function. However, we could replace X with any function and it would be convolved with the exponential. In fact, this is a simple one pole digital filter.

CONVOLUTION OF SAMPLED SIGNALS

In dealing with discrete functions, there are actually two types of problems: (1) the discrete functions are sequences of numbers, or (2) the discrete functions are sampled versions of continuous functions. The key point separating the two is whether or not the time interval between samples is an issue. In the previous section, we treated the first problem; now we will look at the second.

In Fig. 3, suppose we started with continuous functions $x(t)$, $h(t)$, and $y(t)$ instead of $x(n)$, $h(n)$, and $y(n)$, respectively. The convolution in the time domain is

$$y(t) = \int_0^{\infty} h(\tau) \cdot x(t-\tau)\, d\tau \tag{20}$$

where it has once again been assumed that the system response $h(t)$ is causal. Suppose that in order to simulate it on a computer, this problem had to be implemented in the discrete domain. The finite difference approximation of the integral in Eq. (20) is

$$\begin{aligned} y(n) &\cong \sum_{i=0}^{n} h(n-i) \cdot x(i) \cdot T \\ &= T \sum_{i=-\infty}^{n} h(n-i) \cdot x(i) \end{aligned} \tag{21}$$

where T is the time interval between samples. Taking the Z transform of both sides

$$\sum_{n=0}^{\infty} y(n)z^{-n} = T \sum_{n=0}^{\infty} \sum_{i-0}^{n} h(n-i) \cdot x(i) z^{-n}$$

the development becomes identical to the previous section, except we obtain the extra T:

$$Y(z) = T \cdot X(z) \cdot H(z) \tag{22}$$

Simulation of a Two Pole Digital Filter

In this section, we will design a digital filter equivalent to the RLC circuit in Fig. 5(a). It will be convenient to start in the frequency domain [Fig. 5(b)], from which we obtain the following transfer function:

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1/j\omega C}{j\omega L + 1/j\omega C + R} = \frac{1/LC}{1/LC + j\omega R/L - \omega^2} \tag{23}$$

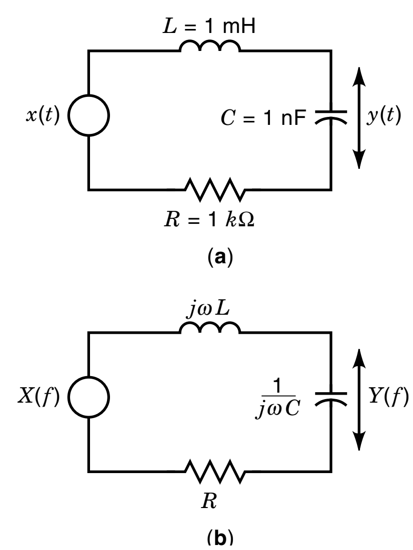

Figure 5. (a) An RLC circuit; (b) Fourier components of the RLC circuit.

To get this into a recognizable form, we will use the following change of parameters:

$$\alpha = \frac{R}{2L} = 0.5 \times 10^6 \qquad \beta = \sqrt{\frac{1}{LC} - \alpha^2} = 0.866 \times 10^6$$
$$\gamma\beta = \frac{1}{LC} \Rightarrow \gamma = 1.155 \times 10^6$$

The Z transform can now be read from the frequency domain expression in Table 1

$$\begin{aligned} H(z) &= Z\left[\gamma \frac{\beta}{(\alpha^2+\beta^2) + j2\alpha\omega - \omega^2}\right] \\ &= 1.155 \times 10^6 \frac{e^{-\alpha T}\sin(\beta \cdot T)\cdot z^{-1}}{1 - 2e^{-\alpha T}\cdot\cos(\beta\cdot T)\cdot z^{-1} + e^{-2\alpha T}} \end{aligned} \tag{24}$$

At first, we may be somewhat startled to see the magnitude of the multiplier resulting from the γ term. But remember, when it is convolved with another function, it will be multiplied by T! Since $\beta = 0.866 \times 10^6$, we will want T to be much smaller, so choose $T = 10^{-7}$. Notice now that

$$e^{-\alpha \cdot T} = e^{-0.05} = 0.951$$
$$e^{-2\alpha \cdot T} = e^{-.1} = 0.904$$

and

$$\sin(\beta \cdot T) = \sin(0.086) = 0.086$$
$$\cos(\beta \cdot T) = \cos(0.086) = 0.9963$$

Now take the convolution of $h(n)$ with an unknown function $x(n)$.

$$\begin{aligned} Y(z) &= H(z)X(z)T \\ &= \frac{1.155\times10^6\cdot(0.951)\cdot(0.086)\cdot10^{-7}}{1 - 2\cdot(0.951)\cdot(0.9963)\cdot z^{-1} + (0.904)z^{-2}} z^{-1}X(z) \\ &= \frac{0.0094}{1 - 1.895z^{-1} + 0.9044z^{-2}} z^{-1}X(z) \end{aligned} \tag{25}$$

$$\begin{aligned} Y(z) &= 1.895\cdot z^{-1}Y(z) - 0.9044z^{-2}Y(z) + 0.0094\cdot z^{-1}X(z) \\ y(n) &= 1.895\cdot y(n-1) - 0.9044\cdot y(n-2) + 0.0094\cdot x(n-1) \end{aligned} \tag{26}$$

Note that the z^{-1} in the numerator of $H(z)$ in Eq. (25) resulted in the $x(n - 1)$, that is, a delay in the input in Eq. (26). The computer code in Fig. 6 implements Eq. (26).

Alternatively, we may be asked to build the digital equivalent of the analog filter shown in Fig. 5. This is done in Fig. 7. Note the use of delay registers (marked D), which hold the values for one clock cycle, changing $y(n)$ to $y(n - 1)$ and $y(n - 1)$ to $y(n - 2)$. These, along with the input, are multiplied by their respective scaling factors and summed to give the new value of $y(n)$ at every clock cycle, as per Eq. (26).

```
T = 1E−7
GAMMA = 1.155e6
DO N=0,NMAX
    Y(N) = 2*EXP(−ALPHA*T)*COS(BETA*T)* Y(N−1)
    − EXP(−2*ALPHA*T) * Y(N−2)+ T*GAMMA*SIN(BETA*T)* X(N−1)
END DO
```

Figure 6. Simulation of the *RLC* circuit in Fig. 5. $X(N)$ is the (as yet unspecified) input.

Sum of Two Parallel Systems

A system diagram is given in Fig. 8. The two transfer functions are

$$H_1(\omega) = \frac{1}{j\omega + \alpha_1} \tag{27}$$

and

$$H_2(\omega) = \frac{1}{j\omega + \alpha_2} \tag{28}$$

Suppose we want to design a digital simulation of this system. The overall transfer function of the system is given by

$$Y(\omega) = [H_1(\omega) + H_2(\omega)]\cdot X(\omega) \tag{29}$$

Going to the Z domain gives

$$\begin{aligned} Y(z) &= [H_1(z) + H_2(z)]\cdot X(z)\cdot T \\ &= \left[\frac{1}{1 - e^{-\alpha_1 T}z^{-1}} + \frac{1}{1 - e^{-\alpha_2 T}z^{-1}}\right]\cdot T\cdot X(z) \\ &= \left[\frac{2 - (e^{-\alpha_1 T} + e^{-\alpha_2 T})z^{-1}}{1 - (e^{-\alpha_1 T} + e^{-\alpha_2 T})z^{-1} + z^{-2}}\right]\cdot T\cdot X(z) \end{aligned} \tag{30}$$

from which we get

$$\begin{aligned} Y(z) &= (e^{-\alpha_1 T} + e^{-\alpha_2 T})z^{-1}Y(z) - z^{-2}Y(z) \\ &\quad + 2\cdot T\cdot X(z) - T\cdot(e^{-\alpha_1 T} + e^{-\alpha_2 T})z^{-1}X(z) \end{aligned} \tag{31}$$

Note that two terms of the input are used with $X(z)$ corresponding to $x(n)$, and $z^{-1}X(z)$ corresponding to $x(n - 1)$. This does not present any particular difficulty.

Going back to Eq. (30), instead of cross multiplying, suppose we define two auxiliary functions

$$S_1(z) = \frac{T}{1 - e^{-\alpha_1 T}z^{-1}}\cdot X(z)$$
$$S_2(z) = \frac{T}{1 - e^{-\alpha_2 T}z^{-1}}\cdot X(z)$$

So instead of the results of Eq. (31), we get

$$S_1(z) = e^{-\alpha_1 T}z^{-1}S_1(z) + T\cdot X(z) \tag{32a}$$
$$S_2(z) = e^{-\alpha_2 T}z^{-1}S_2(z) + T\cdot X(z) \tag{32b}$$
$$Y(z) = S_1(z) + S_2(z) \tag{32c}$$

The results of Eqs. (32) present a simpler formulation. This is a method of defining auxiliary parameters so that several small calculations are being made instead of one large one, often an easier process. If, for instance, H_1 and H_2 were each second-order systems, the cross multiplication similar to Eq. (31) would produce a fourth order system. It would be far better to define two-second order auxiliary parameters and make

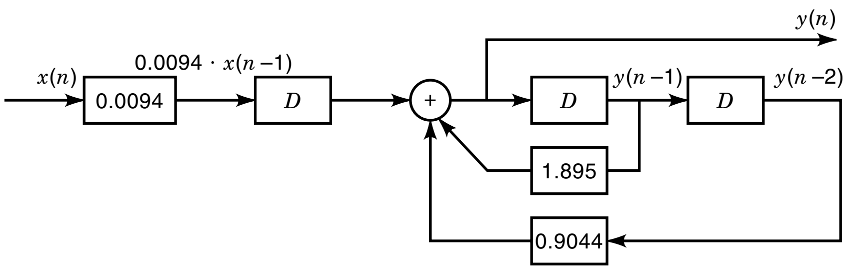

Figure 7. Digital hardware implementation of the analog filter in Fig. 5. The boxes marked D represent delay registers.

two second-order calculations, similar to Eq. (32a) and Eq. (32b). Figure 9 is the digital simulation of the transfer function.

PROPERTIES OF *Z* TRANSFORMS

The definition of the Z transform has been given along with some examples of how the Z transform may be used. In solving these examples, we used the important convolution theorem. Here are some other important properties of Z transforms.

Linearity

If $F_1(z) = Z[f_1(n)]$ and $F_2(z) = Z[f_2(n)]$, then $Z[\alpha f_1(n) + \beta f_2(n)] = \alpha F_1(z) + \beta F_2(z)$

Proof

$$Z[\alpha \cdot f_1(n) + \beta \cdot f_2(n)]$$
$$= \sum_{n=-\infty}^{\infty} (\alpha \cdot f_1(n) + \beta \cdot f_2(n)) z^{-n}$$
$$= \alpha \sum_{n=-\infty}^{\infty} f_1(n) + \beta \sum_{n=-\infty}^{\infty} f_2(n) = \alpha \cdot F_1(z) + \beta \cdot F_2(z)$$

Time-Shifting

If $F(z) = Z[f(n)]$, then $Z[f(n - m)] = z^{-m}F(z)$

Proof

$$Z[f(n-m)] = \sum_{n=-\infty}^{\infty} f(n-m) z^{-n}$$
$$= z^{-m} \sum_{n=\infty}^{\infty} f(n-m) z^{-(n-m)}$$
$$= z^{-m} \sum_{l=-\infty}^{\infty} f(l) z^{-(l)} = z^{-m} F(z) \quad l = n - m$$

Note that this proof was for the general case of a noncausal function $f(n)$. If $f(n)$ is causal, or of a finite duration, then additional terms may appear (3).

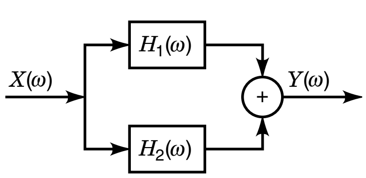

Figure 8. A system made up of two parallel sub systems.

Initial Value Theorem

If $F(z) = Z\,[f(n)]$, then $f(0) = \lim_{z \Rightarrow \infty} F(z)$

Proof The proof comes directly from the definition of the Z transform in Eq. (1). As $z \Rightarrow \infty$, all terms vanish except $f(0)$, which proves the theorem.

Final Value Theorem

If $F(z) = Z[f(n)]$, and $(z - 1)F(z)$ has no poles on or outside the unit circle (*see the section on stability*), then

$$f(\infty) = \lim_{z \to 1} (z - 1) F(z)$$

This theorem can be extremely helpful, because it gives us the steady state value without solving for the entire sequence in the time domain. However, the proof is nontrivial. The interested reader should see Ref. 3.

There are numerous other theorems, some of which are listed in Table 2. More extensive lists are available in the literature (1–4).

THE INVERSE *Z* TRANSFORM

Like all frequency domain transforms, there is an inverse Z transform given by

$$f(n) = \frac{1}{2\pi j} \int_{\Gamma} F(z) z^{n-1} \, dz \tag{33}$$

This involves contour integration in the complex plane and is rarely of practical use (2). Instead, we will use much the same approach we used to get the forward Z transform: get the ex-

```
DO N=0,NMAX
      S1(N) = EXP(−ALPHA1*T)*S1(N−1) + T*X(N)
      S2(N) = EXP(−ALPHA2*T)*S2(N−1) + T*X(N)
      Y(N)  = S1(N) + S2(N)
END DO
```

Figure 9. Computer program to calculate the response of the system in Fig. 8. The responses S1 and S2 are the responses of the two parallel paths in Fig. 8.

Table 2. Properties of the Z Transform

	Sampled Time Domain	Z Domain
	$f(n)$	$F(z)$
Linearity	$\alpha \cdot f(n) + \beta \cdot g(n)$	$\alpha \cdot F(z) + \beta \cdot F(z)$
Time shift	$f(n - m)$	$z^{-m}F(z)$
Initial value	$f(0)$	$\lim_{z\to 0} F(z)$
Final value	$f(\infty)$	$\lim_{z\to 1}(z - 1) \cdot F(z)$
Integration	$\sum_{n=0}^{N} f(n)$	$\frac{1}{1 - z^{-1}} F(z)$
Convolution	$\sum_{n=0}^{\infty} f(n)h(m - n)$	$F(z) \cdot H(z)$
Complex convolution	$f(n) \cdot g(n)$	$\frac{1}{2\pi j}\oint_C G(v)F\left(\frac{z}{v}\right) v^{-1}\,\delta v$

pression in a form we recognize, then look it up in a table. This involves the well-known partial fraction expansion.

Partial Fraction Expansion

Recall the previous example in which we were convolving two sequences, one an exponentially decaying function of Eq. (11)

$$h(n) = Ae^{-nT/t_0} \qquad n = 0, 1, 2, 3, \ldots \tag{11}$$

and the other the discretized unit step function of Eq. (12)

$$u(n) = 1 \qquad n = 0, 1, 2, 3, \ldots \tag{12}$$

Their Z transforms are expressed in Eq. (13)

$$Z[h(n)] = H(z) = \frac{A}{1 - e^{-T/t_0}z^{-1}} \tag{13}$$

and

$$Z[u(n)] = U(z) = \frac{1}{1 - z^{-1}}$$

The desired convolution is

$$Y(z) = H(z)U(z) = \frac{A}{1 - e^{-T/t_0}z^{-1}} \cdot \frac{1}{1 - z^{-1}} \tag{34}$$

To get the inverse Z transform, it is preferable to work with z instead of z^{-1}, so multiply the numerator and denominator by z^2:

$$Y(z) = \frac{A}{z - e^{-T/t_0}} \cdot \frac{1}{z - 1} z^2$$

For reasons that will be apparent a little later, divide both sides by z:

$$\frac{Y(z)}{z} = \frac{A}{z - e^{-T/t_0}} \cdot \frac{1}{z - 1} z \tag{35}$$

Take the partial fraction expansion of $Y(z)/z$:

$$\frac{Y(z)}{z} = \frac{B}{z - e^{-T/t_0}} + \frac{C}{z - 1} \tag{36}$$

Now solve for B and C by multiplying through by their respective denominators and evaluating at the location of the pole in the Z domain:

$$B = (z - e^{-T/t_0})\left[\frac{1}{z - e^{-T/t_0}} \cdot \frac{Az}{z - 1}\right]_{z=e^{-T/t_0}} = \frac{Ae^{-T/t_0}}{e^{-T/t_0} - 1} \tag{37a}$$

$$C = (z - 1)\left[\frac{1}{z - e^{-T/t_0}} \cdot \frac{Az}{z - 1}\right]_{z=1} = \frac{A}{1 - e^{-T/t_0}} \tag{37b}$$

Putting these values in Eq. (36), we arrive at

$$\frac{Y(z)}{z} = \left[\frac{1}{z - 1} - \frac{e^{-T/t_0}}{z - e^{-T/t_0}}\right]\frac{A}{1 - e^{-T/t_0}} \tag{38}$$

The reason for dividing by z before we did the expansion is now apparent: we can multiply both sides of Eq. (38) by z, and on the right side, we can divide the numerators and denominators by z giving

$$Y(z) = \left[\frac{1}{1 - z^{-1}} - \frac{e^{-T/t_0}}{1 - e^{-T/t_0}z^{-1}}\right]\frac{A}{1 - e^{-T/t_0}}$$

Now all the Z terms are in a format that exist in Table 1, and we get

$$\begin{aligned} y(n) &= \frac{A}{1 - e^{-T/t_0}}[1 - e^{-T/t_0}e^{-nT/t_0}]u(n) \\ &= \frac{A}{1 - e^{-T/t_0}}[1 - e^{-(n+1)T/t_0}]u(n) \end{aligned} \tag{39}$$

Partial Fraction Expansion of Multiple Roots

If multiple roots exist at one location, a modification to the partial fraction expansion is needed. Suppose we have

$$\begin{aligned} \frac{Y(z)}{z} &= \left[\frac{N(z)}{(z - p_1)(z - p_2)^r}\right] \\ &= \frac{k_1}{z - p_1} + \frac{k_{21}}{(z - p_2)} + \frac{k_{22}}{(z - p_2)^2} + \cdots + \frac{k_{2r}}{(z - p_2)^r} \end{aligned}$$

The term k_1 is calculated as explained above. The other terms are calculated by the formula

$$k_{rj} = \frac{1}{(r - j)!}\frac{d^{r-j}}{dz^{r-j}}[(z - p_2)^r Y(z)]_{z=p_2}$$

As an example, suppose we convolve the one pole filter of the previous example with a ramp function,

$$x(n) = nu(n)$$

Convolving this with $h(n)$, and going to the Z domain gives

$$Y(z) = \frac{A}{1 - e^{-T/t_0}z^{-1}} \cdot \frac{z^{-1}}{(1 - z^{-1})^2} T \tag{40}$$

or

$$\frac{Y(z)}{z} = \frac{A}{z - e^{-T/t_0}} \cdot \frac{1}{(z-1)^2} Tz$$
$$= \frac{k_1}{z - e^{-T/t_0}} + \frac{k_{22}}{(z-1)^2} + \frac{k_{21}}{z-1}$$
$$k_1 = \left.\frac{ATz}{(z-1)^2}\right|_{z=e^{-T/t_0}} = \frac{ATe^{-T/t_0}}{(e^{-T/t_0}-1)^2}$$
$$k_{22} = \left.\frac{ATz}{(z-e^{-T/t_0})}\right|_{z=1} = \frac{AT}{(1-e^{-T/t_0})} = \frac{AT(1-e^{-T/t_0})}{(1-e^{-T/t_0})^2}$$
$$k_{21} = \frac{d}{dz}\left[\frac{ATz}{(z-e^{-T/t_0})}\right]_{z=1} = AT\left[\frac{(z-e^{-T/t_0}) - z(1)}{(z-e^{-T/t_0})^2}\right]_{z=1}$$
$$= \frac{-ATe^{-T/t_0}}{(1-e^{-T/t_0})^2}$$

Now the inverse of Eq. (40) is

$$y(n) = \frac{ATe^{-T/t_0}}{(e^{-T/t_0}-1)^2}[(e^{-nT/T_0} - 1) + (e^{-T/t_0} - 1)n]u(n)$$

Cross Multiplication

There is an alternative to the partial fraction expansion method that is often easier to implement, particularly when dealing with complex roots. We start by illustrating its use on an earlier problem and then move to an example with complex roots. Starting with Eq. (36) and cross multiplying the terms in the denominators, we can set this equal to Eq. (35)

$$\frac{Y(z)}{z} = \left[\frac{B(z-1) + C(z-e^{-T/t_0})}{(z-e^{-T/t_0})(z-1)}\right] = \frac{ATz}{(z-e^{-T/t_0})(z-1)}$$

Equating the numerators gives

$$B(z-1) + C(z - e^{-T/t_0}) = ATz$$

and then, by equating like powers of z

$$B = -Ce^{-T/t_0}$$

and

$$C(1 - e^{-T/t_0})z = ATz$$

which leads to the same values as Eq. (37). The advantage to this approach will become apparent with more complicated expressions.

Suppose we go back to the *RLC* filter problem and calculate the expression for the response to a unit step function. Eq. (25) becomes

$$Y(z) = \frac{0.0094z^{-1}}{1 - 1.895\cdot z^{-1} + 0.9044z^{-2}} \frac{1}{1-z^{-1}}$$

Expanding this out into the separate terms, we get the following general form

$$Y(z) = \frac{0.0094z}{z^2 - 1.895\cdot z^1 + 0.9044} \cdot \frac{z}{z-1}$$
$$= \frac{K_1z^2 + K_2z}{z^2 - 1.895\cdot z^1 + 0.9044} + \frac{K_3z}{z-1} \qquad (41)$$

It is best to start by solving for K_3. First divide through by $z/(z-1)$ and solve:

$$K_3 = \left[\frac{0.0094z}{z^2 - 1.895\cdot z^1 + 0.9044}\right]_{z=1}$$
$$= \frac{0.0094}{1 - 1.895 + 0.9044} = \frac{0.0094}{0.0094} = 1$$

Hereafter, however, we are reduced to cross multiplying

$$\frac{0.0094z}{z^2 - 1.895\cdot z + 0.9044} \cdot \frac{z}{z-1}$$
$$= \frac{(K_1z^2 + K_2z)(z-1) + z(z^2 - 1.902\cdot z + 0.9044)}{(z^2 - 1.895\cdot z + 0.9044)(z-1)}$$

Equating like powers of z in the numerator gives

$$0 = K_1z^3 + z^3 \Rightarrow K_1 = -1$$
$$0.0094z^2 = K_2z^2 + K_1z^2 - 1.902\cdot z^2 \Rightarrow K_2 = 0.9044$$

and the accuracy can be checked by the remaining equation

$$0 = -K_2z + 0.9044z$$

Now plugging these numbers back into Eq. (43) gives

$$Y(z) = \frac{0.0094z}{z^2 - 1.895\cdot z^1 + 0.9044} \cdot \frac{z}{z-1}$$
$$= \frac{-z^2 + 0.9044z}{z^2 - 1.895\cdot z^1 + 0.9044} + \frac{z}{z-1}$$
$$= \frac{1}{1-z^{-1}} - \frac{1 - 0.9044z^{-1}}{z - 1.895\cdot z^{-1} + 0.9044z^{-2}}$$
$$= \frac{1}{1-z^{-1}} - \frac{1 - 0.9044z^{-1}}{z - 2\cdot e^{-\alpha T}\cos(\beta T)\cdot z^{-1} + e^{-2\alpha T}z^{-2}}$$

This is starting to look like something from Table 1, but the last term resembling the decaying cosine isn't quite there. The following manipulation is required:

$$Y(z) = \frac{1}{1-z^{-1}} - \frac{1 - 0.9044z^{-1}}{z - 2\cdot e^{-\alpha T}\cos(\beta T)\cdot z^{-1} + e^{-2\alpha T}z^{-2}}$$
$$= \frac{1}{1-z^{-1}} - \frac{1 - e^{-\alpha T}\cos(\beta T)z^{-1}}{z - 2\cdot e^{-\alpha T}\cos(\beta T)\cdot z^{-1} + e^{-2\alpha T}z^{-2}}$$
$$+ \frac{0.521\cdot e^{-\alpha T}\sin(\beta T)z^{-1}}{z - 2\cdot e^{-\alpha T}\cos(\beta T)\cdot z^{-1} + e^{-2\alpha T}z^{-2}}$$

Notice that it was necessary to break the last term into two parts to give two terms that can be found in Table 1

$$y(n) = [1 - e^{-\alpha nT}\cos(\beta nT) - 5.21\cdot e^{-\alpha nT}\sin(\beta nT)] \qquad (42)$$

STABILITY

In dealing with the transfer function $H(z)$ of a discrete system, a key issue is the stability of this system. By stability we are saying that the output remains bounded for any bounded input (3). An Nth order causal system has a transfer function which can be expressed as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_{N-1} z + b_N}{a_0 z^N + a_1 z^{N-1} + \cdots + a_{N-1} z + a_N} \quad (43)$$

In determining the response to an input $X(z)$, the output can be expressed in the following manner after factoring the denominator into its roots and taking the partial fraction expansion

$$Y(z) = \frac{c_1 z}{z - p_1} + \frac{c_2 z}{z - p_2} + \cdots + \frac{c_N z}{z - p_N} + Y_I(z) \quad (44)$$

(For this example, it is assumed there are no repeated roots.) $Y_I(z)$ contains only those terms that originated from the poles of the input $X(z)$. Taking the inverse transform of Eq. (44) gives

$$y(n) = c_1 p_1^n + c_2 p_2^n + \cdots + c_N p_N^n + y_I(n) \quad (45)$$

Each of the poles of Eq. (44) produced an exponential term in Eq. (45). As long as the magnitude of each of the poles in Eq. (44) is less than one, then each of the terms in Eq. (45) is exponentially decaying. The term $y_I(n)$ originated from the input, which we have already assumed is bounded. Therefore, looking at Fig. 10, we can say that the system is stable if each pole is inside the unit circle in the complex Z plane.

The *RLC* filter that we analyzed earlier had the following transfer function:

$$H(z) = 1.155 \times 10^6 \frac{e^{-\alpha T} \sin(\beta \cdot T) \cdot z^{-1}}{1 - 2e^{-\alpha T} \cos(\beta \cdot T) \cdot z^{-1} + e^{-2\alpha T} z^{-2}}$$
$$= \frac{0.0818 z^{-1}}{1 - 1.895 z^{-1} + 0.904 z^{-2}}$$

The denominator has its roots at $z = 0.9475 + j\ .079$ and $0.9475 - j\ .079$, forming a complex conjugate pair. Most important is that $|z| = 0.9508$; both roots have a magnitude of less than one and lie inside the unit circle.

ALTERNATIVE METHODS TO FORMULATE THE *Z* TRANSFORM

We have seen examples in which problems were stated in the frequency domain and we solved them in the sampled time domain. Our approach has been to take the partial fraction expansion of the frequency domain expression, find the corresponding Z transforms, and solve the problem in the Z domain. Our success depended upon the ability to manipulate the frequency domain expression in a form that could be found in a table like Table 1. In this section, we present some alternatives.

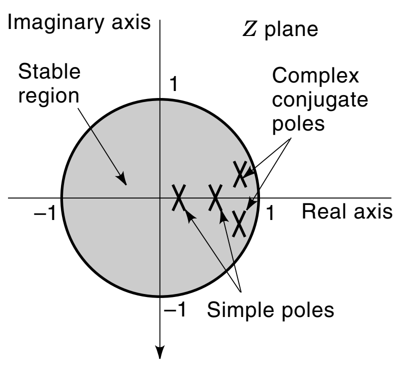

Figure 10. Complex Z plane. All poles must be inside the unit circle (shaded area) to insure stability.

Backward Rectangular Approximation

Fourier Transform Theory tells us that a $j\omega$ in the frequency domain becomes a derivative in the time domain (1,2). In going from the time domain to the sampled time domain, the derivative may be approximated by

$$\frac{df(t)}{dt} \cong \frac{f[nT] - f[(n-1)T]}{T}$$

Taking the Z transform of the right side gives

$$Z\left\{\frac{f[nT] - f[(n-1)T]}{T}\right\} = \frac{F(z) - z^{-1}F(z)}{T} = \frac{1 - z^{-1}}{T} F(z)$$

Suppose this is taken one step further and the transition from the frequency domain to the Z domain is made by simply replacing

$$j\omega \Rightarrow \frac{1 - z^{-1}}{T} \quad (46)$$

As an example, the transition from the frequency domain to the Z domain for the one pole filter becomes

$$\frac{1}{\alpha + j\omega} \Rightarrow \frac{1}{\alpha + \frac{1 - z^{-1}}{T}} = \frac{T}{\alpha T + 1 - z^{-1}} \quad (47)$$

At first glance, this does not seem in any way, shape, or form to represent the Z transform in Table 1. An approximation that is useful here and elsewhere is

$$\frac{1}{1 + \delta} \cong e^{-\delta} \qquad \text{if} \qquad \delta \ll 1$$

Utilizing this approximation Eq. (47) becomes

$$\frac{T}{\alpha T + 1 - z^{-1}} = \frac{T/(1 + \alpha T)}{1 - z^{-1}/(1 + \alpha T)} \cong \frac{T e^{-\alpha T}}{1 - e^{-\alpha T} z^{-1}} \quad (48)$$

There are two points worth noting. First, the factor T that we usually add in the convolution theorem is already there because the substitution of Eq. (48) is essentially Riemann Integration, that is, approximating an integral by a summation at specific intervals of size T. Furthermore, the amplitude has been changed by a factor of

$$\frac{1}{1 + \alpha T} \cong e^{-\alpha T}$$

Once again, if αT is small, this term will be very close to 1. This means that more accuracy can be obtained by making T smaller.

The practical reasoning for this approach is a little clearer if we go back to the *RLC* circuit of Fig. 5, which had the trans-

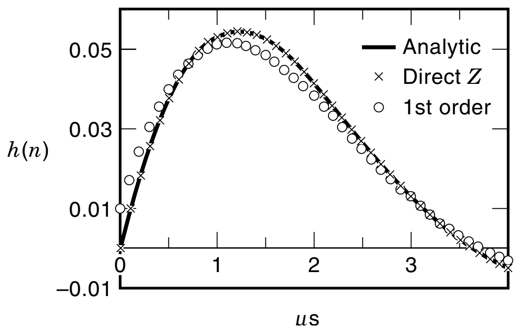

Figure 11. Impulse response of the second-order *RLC* circuit as calculated analytically (dashed curve), by the direct Z transforms (×), and by the first-order backward rectangular approximation (○). Notice that the latter is not as accurate. However, it can be made arbitrarily close to the analytic solution by using smaller time steps.

fer function

$$H(\omega) = \frac{\omega_0^2}{\omega_0^2 + j\omega 2\delta_0\omega_0 - \omega^2}$$

where

$$\omega_0 = 1/LC \qquad \delta_0 = R/2L\omega_0$$

Instead of having to transform this to the form in Table 1, replace each $j\omega$ with $(1 - z^{-1})/T$:

$$\begin{aligned} H(z) &= \frac{\omega_0^2}{\omega_0^2 + \left(\dfrac{1-z^{-1}}{T}\right)2\delta_0\omega_0 + \left(\dfrac{1-z^{-1}}{T}\right)^2} \\ &= \frac{\omega_0^2T^2}{\omega_0^2T^2 + (1-z^{-1})2\delta_0\omega_0T + (1-2z^{-1}+z^{-2})} \\ &= \frac{\omega_0^2T^2}{(\omega_0^2T^2 + 2\delta_0\omega_0T + 1) - 2(1+\delta_0\omega_0T)z^{-1} + z^{-2}} \end{aligned} \tag{49}$$

The resemblance to the Z transform of Table 1 is not as obvious, but it is there. Figure 11 shows the impulse response using Eq. (49) compared to the results obtained from Eq. (26). These plots were made using a time step of 0.1 μs. If the time step is reduced to 0.01 μs, the results correspond almost exactly.

Trapezoidal Approximation (Bilinear Transform)

Equation (46) can be improved upon by the following transformation:

$$j\omega \Rightarrow \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \tag{50}$$

This is referred to as the bilinear transform. While the use of Eq. (46) can be thought of as approximating the time domain convolution integral with rectangular step Reimann integration, Eq. (50) represents trapezoidal integration. Equation (50) is preferred by theoreticians in the signal processing field because it is unconditionally stable, whereas Eq. (46) is not (3). From a somewhat more intuitive view, it is more accurate because trapezoidal integration is more accurate than rectangular integration. The disadvantage is obvious: the order of the system in the Z domain is doubled!

As a simple example, take the Z transform of the one pole function using the bilateral transform (4)

$$\begin{aligned} \frac{1}{\alpha + j\omega} &= \frac{1}{\alpha + \dfrac{2}{T}\dfrac{1-z^{-1}}{1+z^{-1}}} = \frac{(1+z^{-1})\cdot T}{(1+z^{-1})\cdot\alpha T + 2\cdot(1-z^{-1})} \\ &= \frac{(1+z^{-1})\cdot T}{(\alpha T+2) + (\alpha T - 2)z^{-1}} = \frac{(1+z^{-1})\cdot\dfrac{T}{2+\alpha T}}{1 + \dfrac{1-\alpha T/2}{1+\alpha T/2}z^{-1}} \end{aligned} \tag{51}$$

Figure 12 compares the formulation of Eq. (48) with the new bilateral formulation from Eq. (51) and with the analytic formulation of Eq. (22). Clearly Eq. (51) is more accurate, after the first pulse. (Notice that $1 + z^{-1}$ in the numerator of Eq. (51) means that the impulse response is calculated by averaging the impulse over the first two time steps.)

The different approaches used to change from the Fourier domain to the Z domain can be summarized as follows: The direct transform, that is, converting from terms in the frequency domain to those in the Z domain by looking them up in a table, is the most accurate and also gives the lowest order expression in the Z domain. The disadvantage of the direct transform is that it often requires a partial fraction expansion. For third and higher order systems, this is not trivial. Using direct substitution via either the backward approximation or the bilateral transform is usually easier, however, these are approximations. The bilateral transform is better than the backward approximation, at the cost of increased complexity.

Most authors describe these transforms starting from the Laplace domain (3,4). The concepts are the same, but begin with s instead of $j\omega$.

AN EXAMPLE FROM ELECTROMAGNETIC SIMULATION

The past decade has seen a dramatic increase in the use of computer simulation methods for a wide variety of applications in electromagnetics. In particular, time domain methods, such as the finite-difference time-domain (FDTD) method have become more popular because of their flexibility and

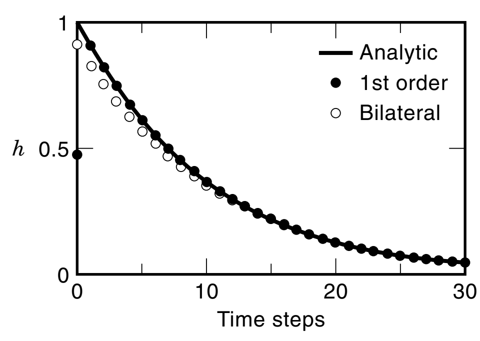

Figure 12. Impulse response of the single-pole filter as calculated analytically (dashed curve), by the first-order backward rectangular approximation (○), and by the second-order bilateral approximation (●). The bilateral is more accurate, but requires twice as many terms.

their efficiency in utilizing the power of today's computers (5). In this example, it will be demonstrated that the Z transform can be used in formulating the FDTD simulation for complex materials (6,7).

The time dependent Maxwell's equations are given by

$$\frac{dD}{dt} = \nabla \times H \tag{52a}$$

$$D(\omega) = \epsilon_0 \cdot \epsilon_r^*(\omega) \cdot E(\omega) \tag{52b}$$

$$\frac{dH}{dt} = \frac{1}{\mu_0} \nabla \times E \tag{52c}$$

where μ_0 is the permeability and ϵ_0 is the permittivity of free space. It will be assumed that we are dealing with nonmagnetic materials. However, the relationship between the flux density D and the electric field E can be an extremely complicated function of frequency.

In implementing the FDTD method, Eqs. (52a) and (52c) are formulated using spatial and temporal difference approximations. This is straightforward and is described extensively in the literature (5). However, we still need a method of calculating E from D. We will regard this as a digital filtering problem, and utilize the Z transforms.

Suppose we are simulating a medium described by the following complex dielectric constant

$$\epsilon_r^*(\omega) = \epsilon_r + \frac{\sigma}{j\omega\epsilon_0} + \epsilon_1 \frac{\omega_0}{(\omega_0^2 + \alpha^2) + j2\alpha\omega + \omega^2} \tag{53}$$

Inserting Eq. (53) into Eq. (52b) and taking the Z transforms, we get

$$\begin{aligned} D(z) = \epsilon_r E(z) &+ \frac{\sigma \cdot T/\epsilon_0}{1 - z^{-1}} \cdot E(z) \\ &+ \epsilon_1 \frac{e^{-\alpha T} \cdot \sin(\omega_0 T) \cdot T \cdot z^{-1}}{1 - 2e^{-\alpha T}\cos(\omega_0 T) \cdot z^{-1} + e^{-2\alpha T} z^{-2}} \cdot E(z) \end{aligned} \tag{54}$$

It will prove worthwhile to define two auxiliary parameters:

$$I(z) = \frac{\sigma \cdot T/\epsilon_0}{1 - z^{-1}} \cdot E(z) \tag{55a}$$

$$S(z) = \frac{\epsilon_1 \cdot e^{-\alpha T} \cdot \sin(\omega_0 T) \cdot T \cdot E(z)}{1 - 2e^{-\alpha T}\cos(\omega_0 T) \cdot z^{-1} + e^{-2\alpha T} z^{-2}} \tag{55b}$$

Now Eq. (54) becomes

$$D(z) = \epsilon_r E(z) + I(z) + z^{-1} S(z) \tag{56}$$

Once we have calculated $E(z)$, $I(z)$ and $S(z)$ can be calculated from Eqs. (55a) and (55b):

$$I(z) = z^{-1} I(z) + \frac{\sigma \cdot T}{\epsilon_0} E(z) \tag{57a}$$

$$\begin{aligned} S(z) &= 2e^{-\alpha T}\cos(\omega_0 T) \cdot z^{-1} S(z) \\ &\quad - e^{-2\alpha T} z^{-2} S(z) + \epsilon_1 \cdot e^{-\alpha T} \cdot \sin(\omega_0 T) \cdot T \cdot E(z) \end{aligned} \tag{57b}$$

The trouble is this: in calculating $E(z)$ in Eq. (56), we need the previous value of $S(z)$, the present value of $D(z)$, and the present value of $I(z)$. The present value of $D(z)$ is not a problem because in the order in which the algorithm is implemented, it has already been calculated in Eq. (52a). However, the present value of $I(z)$ requires the present value of $E(z)$. This problem is circumvented by simply replacing $I(z)$ with its expanded version in Eq. (57a). Now Eq. (56) becomes

$$D(z) = \epsilon_r E(z) + z^{-1} I(z) + \frac{\sigma \cdot T}{\epsilon_0} \cdot E(z) + z^{-1} S(z)$$

from which $E(z)$ can be calculated by

$$E(z) = \frac{D(z) - z^{-1} I(z) - z^{-1} S(z)}{\epsilon_r + \sigma \cdot T/\epsilon_0} \epsilon_r E(z) \tag{58}$$

$$I(z) = z^{-1} I(z) + \frac{\sigma \cdot T}{\epsilon_0} \cdot E(z) \tag{57a}$$

$$\begin{aligned} S(z) &= 2e^{-\alpha T}\cos(\omega_0 T) \cdot z^{-1} S(z) \\ &\quad - e^{-2\alpha T} z^{-2} S(z) + \epsilon_1 \cdot e^{-\alpha T} \cdot \sin(\omega_0 T) \cdot T \cdot E(z) \end{aligned} \tag{57b}$$

Note that $S(z)$ did not have to be expanded out because it already had a z^{-1} in the numerator.

SUMMARY

The Z transform plays the same role in discrete time that the Laplace and Fourier transforms play in continuous time. As shown earlier in this article, it can be used to analyze discrete time equations, develop iterative solutions to discrete time equations, or design digital circuitry to calculate a solution in hardware. It has even found application in electromagnetic simulation.

The theory and examples in this article are by no means complete. However, they illustrate the power and flexibility of the Z transform as one of the essential tools in electrical engineering today.

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