CHAOS TIME SERIES ANALYSIS

Chaos theory is a relatively new concept. It was first introduced in the 1970s with applications in meteorology (1). Since that time, chaos theory and fractal analysis have been applied in numerous areas in addition to meteorology, including medicine, economics, and the social sciences (2,3).

The basic concept underlying chaotic systems is that seemingly simple equations can produce extremely complex behavior when followed over time. One of the earliest equations studied was the logistic equation, sometimes called the Poincaré equation. This equation is in the form of a recurrence relation, in which subsequent terms are some combination of previous terms. For example, in the logistic equation the $(n + 1)$th term in the sequence is given in terms of the nth term:

$$a_{n+1} = Aa_n(1 - a_n) \tag{1}$$

where A is a constant chosen in the range $2 \leq A \leq 4$, and n is considered to be a point in time. Note that most researchers and textbooks assume that n takes integer values. Later we will discuss Eq. (1) in the context of real values of n. The solution viewed under real value conditions corresponds more closely to experimental data.

This equation has one initial, or boundary, condition a_0, $0 \leq a_0 \leq 1$. Figure 1 shows a bifurcation map of the logistic equation. For values of A close to 2, the sequence converges to a single value. As the value of A increases, the sequence bifurcates (that is, it oscillates between two values). These bifurcations themselves then bifurcate for higher values of A. When A exceeds 3.57, all values begin to fill in, an indication of the onset of chaos. In fact, Eq. (1) displays the stan-

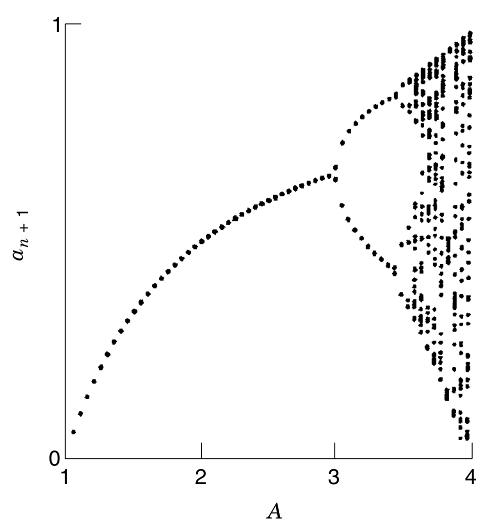

Figure 1. Bifurcation map for the logistic equation showing progression from single point convergence to chaos.

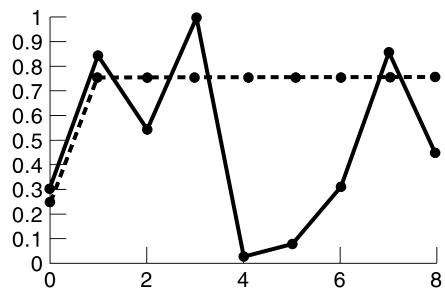

Figure 2. Plot of logistic equation for $a_0 = 0.25$ and $a_0 = 0.3$ demonstrating sensitivity to initial conditions.

dard characteristics of a chaotic equation if the constant A exceeds the value of 3.57. These conditions include

1. Lack of periodic pattern that repeats itself at regular intervals
2. Sensitivity to initial conditions (i.e., a small change in the value of a_0 results in a large change in long-term behavior). Thus two points that are close to each other at the beginning ($n = 0$) may have significantly different values when n is large.

Figure 2 illustrates the second condition. Two plots of n versus a_n, with different initial conditions, $a_0 = 0.25$ and $a_0 = 0.3$, $A = 4$, result in a dramatic change in the subsequent values of a_n. For this example, the difference is quite dramatic. For an initial value of $a_0 = 0.25$, the equation in fact converges, representing a point of stability even in the region of chaos. On the other hand, an initial value of $a_0 = 0.3$ results in chaos.

The logistic equation is an example of a seemingly simple equation that results in chaos. Many nonlinear differential equations also result in chaos. The chaotic nature of these equations poses a major problem in modeling many physical processes, as these models rely on nonlinear models. Applications that result in nonlinear dynamic models include fluid dynamics, meteorology, and hemodynamics. As the models for these systems usually have mathematical formulations that cannot be solved analytically (1), numerical methods that involve discrete solutions of the continuous models are employed. These discrete solutions are similar in nature to Eq. (1) and thus pose serious problems due to the consequences of chaotic systems.

One problem that arises in the aforementioned modeling process is determining whether the systems themselves are chaotic or whether the discrete numerical solution introduces chaotic factors that are not present in the original continuous model. Insight into this problem can be gained through examination of the solution of Eq. (1), which in fact has an exact solution for the values $A = 2$ and $A = 4$. It is also possible to obtain an approximate solution for other values of A. It will be illustrated later that these solutions change the perception of chaos in this system (4).

An important topic that is connected to chaos is the concept of fractals. Fractals have the characteristic of repetition of the same pattern on different scales (5). Figure 3 illustrates the concept of fractals in a geometrical sense. This image is called the Koch snowflake. The large triangle (a) is divided into smaller triangles in an infinite progression. Part (b) shows the initial division, and part (c) the beginning of the next division.

In nature, the idea of a fractal can be seen if one looks at a structure, such as a coastline on a map (6). Depending on the scale of the map, the details of the coastline will change. If a map of the state of California is considered, a few major curves may be noted that indicate San Francisco Bay, Monterey Bay, and so on. If, however, a map of San Francisco is considered, many more details are seen in San Francisco Bay, which itself contains many smaller indentations. This process can be continued down to the scale of grains of sand. Thus the length of the coastline depends on the length of our measuring device. This process leads to the concept of fractal dimension, which in short is the measure of irregularity of a surface. It is an intermediate measure that extends the idea of one, two, or three dimensions to include fractional dimensions. Thus a fractal dimension is any noninteger dimension.

The idea of fractal dimension is also important in measuring the irregularity of mathematical constructs that may not be geometric in nature. Thus we can model a system and look at its fractal dimension to determine the nature of the system in terms of chaotic behavior. Mathematical details of this approach will be given in the next section. In another type of application, fractals have found widespread use in computer graphics to produce realistic images based on patterns repeating on different geometric scales.

A major area of work that utilizes chaos theory is the modeling of nonlinear dynamical systems. Chaos theory is attractive for these models because traditional mathematical models cannot be solved analytically (1). One of the areas of greatest interest is in cardiology applications, where a number of chaotic techniques have been employed to model the functioning of the heart and circulatory system (7). Models are used both to explain the system at a functional level and to classify patients according to presence or absence of disease. The goal is to determine the degree of chaos, or variability, that is present. The classification process is based on the identification of chaos, or the degree of chaos, associated with specific diseases. Typically the process involves physical measures of system behavior, such as electrocardiograms (ECG), which are time-series data showing the functioning of the heart through the interpretation of electrical signals.

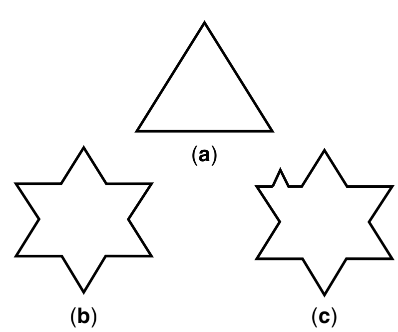

Figure 3. A geometric example of fractals illustrating the repetition of a pattern on different scales: the Koch snowflake.

A number of techniques are available that can show either graphically or numerically the degree of chaos in a system. Graphical methods include logistic maps, Poincaré plots, second-order difference plots, and strange attractors. Logistic maps are another method for viewing the onset of chaos that are related to the bifurcation map shown in Fig. 1. An example is given in the next section. Poincaré plots graph a_{n+1} versus a_n. A plot with points clustering close together indicates little chaotic behavior, while a wide dispersion of points shows a high level of chaos. Second-order difference plots graph $a_{n+2} - a_{n+1}$ versus $a_{n+1} - a_n$ and are similar to Poincaré plots except that the points are centered on the origin. These graphs can be used for both theoretical models and experimental data. Strange attractors are also used as indicators of chaos and have fractal dimensions. These approaches require some mathematical foundation and will be discussed later.

Numerical methods for evaluating the chaotic nature of a system include fractal dimension, Lyapunov exponent, and the central tendency method (CTM). The higher the fractal dimension, the more chaos is present in the system. The fractal dimension can be used to determine the dimension of strange attractors. The Lyapunov exponent is used in the same way and is related to the fractal dimension. The central tendency measure is used as a numerical summary of the second-order difference plot. A low value indicates a high degree of chaos (few points are centered near the origin). Mathematical details for these methods are given in the following sections.

METHODS FOR EVALUATION OF TIME SERIES DATA

Graphical Representations of Chaos

Bifurcation Maps. Referring to the bifurcation map in Fig. 1, it is easy to see the progression from convergence to bifurcation to chaos. Although this map is for the logistic equation, similar maps can be drawn for other chaotic equations. The logistic map offers another method for viewing the same phenomenon.

Logistic Maps. Using Eq. (1), logistic maps also can be drawn that use the differential equation analog of the difference equation:

$$dx/dt = Ax(1 - x) \qquad (2)$$

The logistic map consists of three parts:

1. A parabolic curve $y = Ax\,(1 - x)$
2. A diagonal line $x_{n+1} = x_n$
3. A set of lines connecting the value at x_n with the value at x_{n+1}

The set of lines is repeated until a steady state is reached. Figure 4 shows this graph for $A = 2$ and initial condition $x_0 = 0.3$. If the system exhibits chaotic properties, the logistic map will be characterized by boxes that begin to cover the entire space (8).

Strange Attractors. An attractor is a point of convergence. Attractors are often used in phase space. The phase space is defined for a particle moving in one dimension as its position

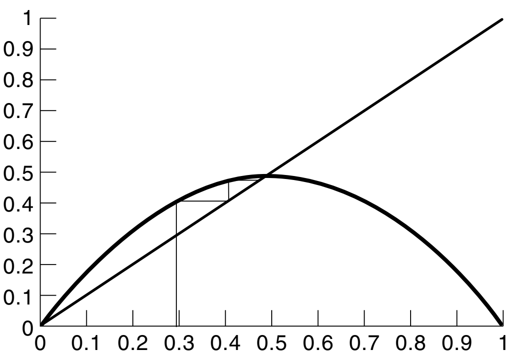

Figure 4. Logistic map for $A = 2$, $a_0 = 0.3$, in the logistic equation which illustrates a nonchaotic state.

x and its velocity v; thus its phase space is a plane. Similarly, a particle moving in two dimensions would have a four-dimensional phase space: two for position and two for velocity. If a phase space is conservative (constant energy), then all points in a given area of phase space at one point in time move so that the area occupied by these points remains constant at later points in time. In three-dimensional systems, the areas becomes a volume, with appropriate generalizations to higher dimensions.

Dynamical systems can be classified as conservative or dissipative. In dissipative systems the phase areas contract. Equations of motion can be determined for conservative and dissipative systems. Consider a set of phase points in a box of volume V. In time dt the volume will change to $V + dV$. Using the product rule and approximating the result with a Taylor series yields the logarithm volume change (9):

$$(1/V)dV/dt = \nabla F \qquad (3)$$

where F represents the equation of motion. If ∇F is zero, the system is conservative; if it is negative, the system is dissipative.

A contractive space representing the phase-space trajectories of the damped pendulum is shown in Fig. 5, with areas contracting to a point. This point is said to be an attractor (10). In chaotic systems, the concept of attractors becomes more complex. Instead of the type of attractor we see in Fig. 5, which is in two-dimensional space, a chaotic attractor has

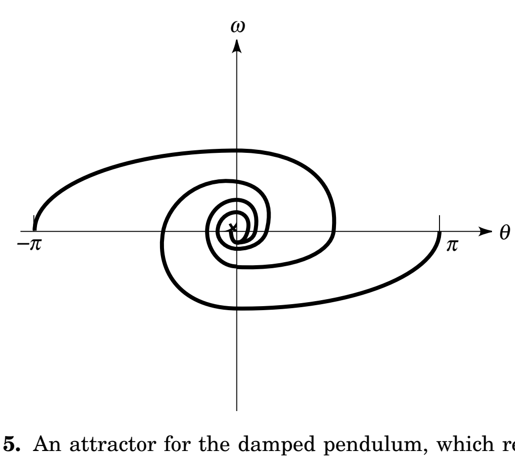

Figure 5. An attractor for the damped pendulum, which represents a contractive space.

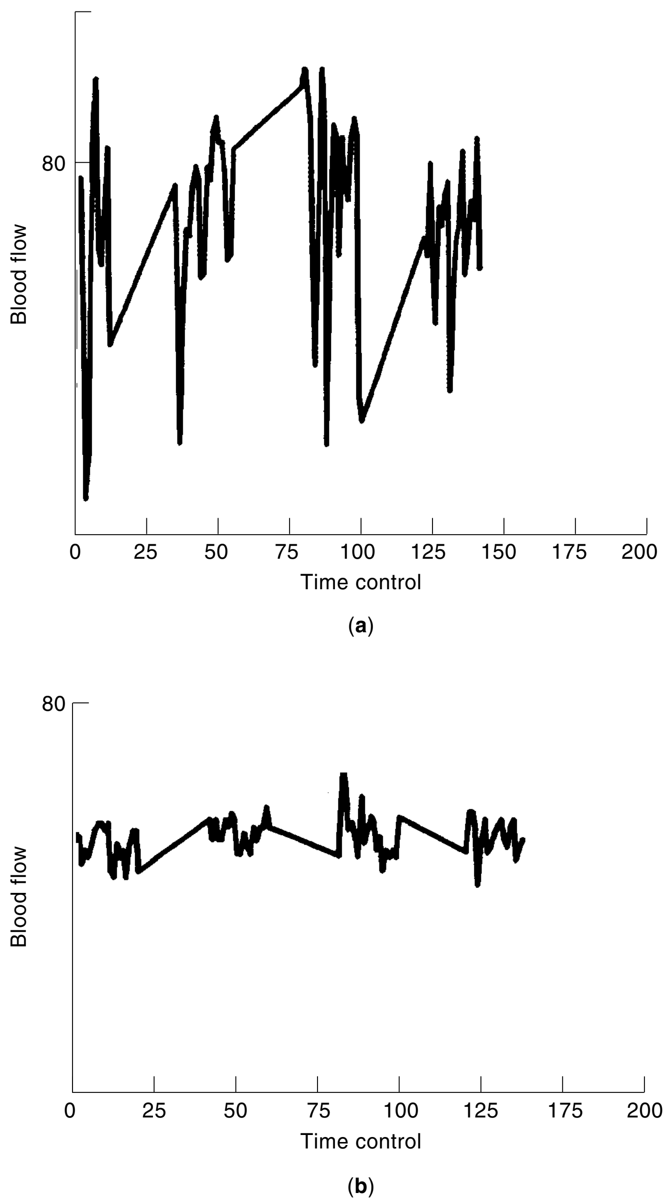

Figure 6. Time series for hemodynamic studies. (a) Control (high variability), (b) nicotine (low variability).

a fractal dimension greater than two. An attractor that has a noninteger dimension (i.e., a fractal dimension) is called a strange attractor. The presence of a strange attractor indicates the presence of chaos. In general, low-dimensional attractors are embedded in lower-dimensional space. As the actual dimension increases, it is necessary to have more points available to achieve representative results. This often limits the usefulness of this approach in dealing with experimental data sets (11).

A Poincaré section simplifies phase-space diagrams for complex systems. It consists of cutting a spiral attractor at regular intervals along the ϕ axis through the (θ, ω) plane, as shown in Fig. 5.

Difference Plots. As mentioned earlier, second-order difference plots are similar in nature to Poincaré plots, but they are centered at the origin and represent the rate of variability. As an example, Fig. 6 shows time series of hemodynamic studies of hepatic blood flow recorded using implanted pulsed Doppler flow meters in an animal model (12). The first time series shows the normal blood flow in the conscious animal. The second time series shows the blood flow after the introduction of nicotine. Note that the variability has decreased significantly in the second time series. Figure 7 shows the second-order difference plots that correspond to these two time series. Sec-

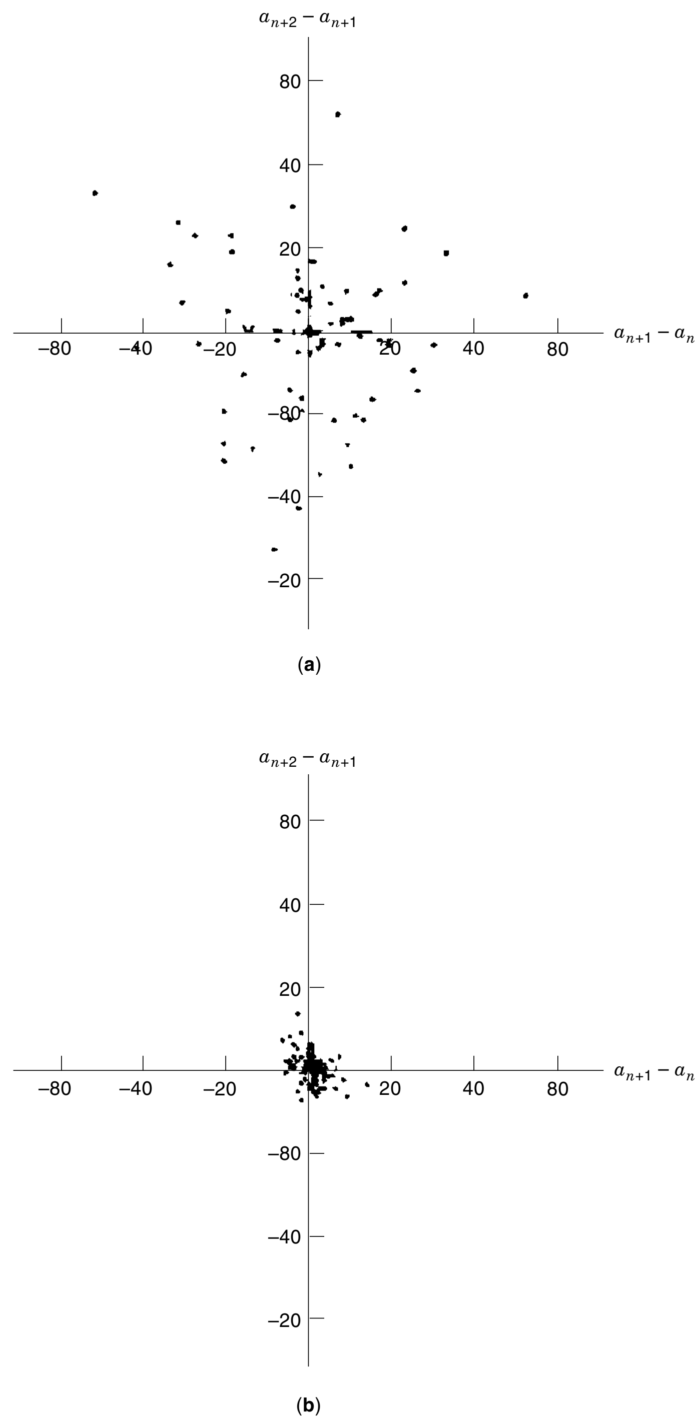

Figure 7. Second-order difference plots corresponding to time series in Fig. 4: (a) control, (b) nicotine.

ond-order difference plots are useful in modeling biological systems, such as hemodynamics and heart rate variations, and represent the degree of theoretical chaos. The difference approach appears to give a more robust picture of the problem and fits well within the theoretical results for the continuous logistic equation. Figure 8 shows theoretical representations for $A = 3.75$, for which a low degree of chaos is seen, and $A = 4.0$, for which a high degree of chaos is seen. Both of these values of A are within the region of chaos. The initial value is $a_0 = 0.5$ for both plots.

Power Spectrum Analysis. A common technique used in the analysis of time series is spectral analysis. Spectral analysis involves the use of the Fourier transform $F(\omega)$ of the time series $f(t)$:

$$F(\omega) = [1/(2\pi)] \int_{-\infty}^{\infty} f(t) \exp(-i\omega t)\, dt \quad (4)$$

The Fourier transform is useful in discerning properties of the time series. In general, periodic functions result in a power spectra that contains one principal component at the drive frequency along with some higher-frequency harmonics.

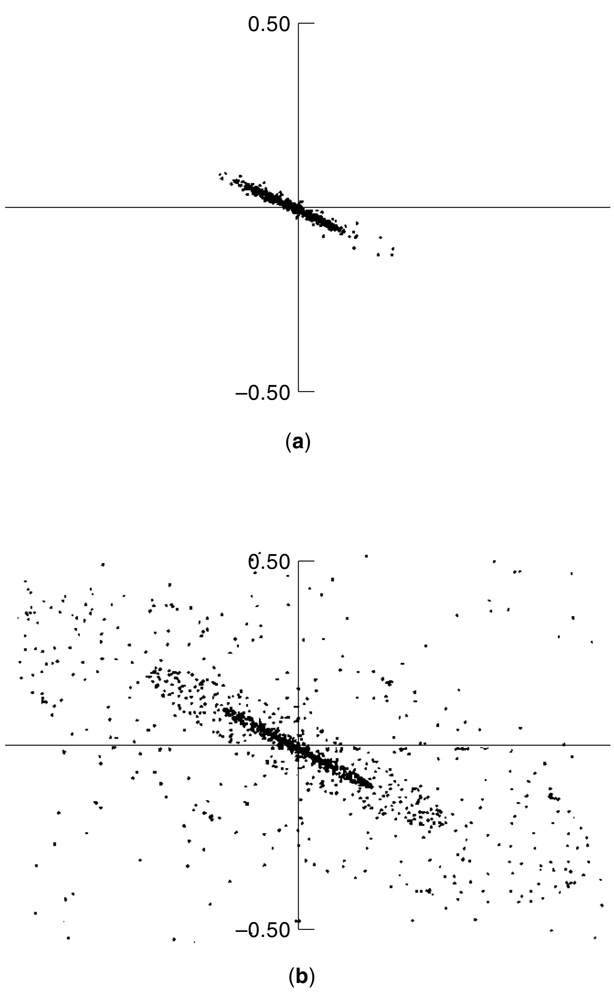

Figure 8. Second-order difference plots for conjectured solution of logistic equation at $A = 3.75$ (low variability) (a) and $A = 4.0$ (high variability) (b).

Chaotic systems, on the other hand, result in a broadband power spectrum with substantial power at low frequencies (13).

Measures of Chaos

While graphical representations are useful for obtaining insight into chaotic systems, the use of numerical methods allows information to be used in more complex decision algorithms for the purposes of classification. A number of numerical measures of chaos have been developed, some of which are described here.

Feigenbaum Number. In a system that exhibits bifurcations, the ratios of spacing between subsequent bifurcations has been found to be constant. This number is called the Feigenbaum number δ such that

$$\lim_{k\to\infty} (A_k - A_{k-1})/(A_{k+1} - A_k) = \delta = 4.669\ldots \quad (5)$$

where A_k is the value at the kth bifurcation. This number applies to maps with a quadratic maximum, such as the logistic map for the logistic equation.

Fractal Dimension. There are a number of interpretations of the fractal dimension. The most common is the capacity dimension, d_c. The approach is to cover a set of points c by volume elements needed to cover c, where c is a subset of the Euclidean space R^n. Let $N(\epsilon)$ denote the smallest number of n-dimensional cubes with sides of length ϵ required to cover c. Then

$$d_c = \lim_{\epsilon\to 0} \log[N(\epsilon)]/\log[1/\epsilon] \quad (6)$$

As an example, consider the Koch snowflake in Fig. 3. It begins with an equilateral triangle. The middle third of each side is then used as the base for a new equilateral triangle. The length of ϵ is reduced by a factor of $\frac{1}{3}$ each time and the number of ϵ's increases by 4, so $\epsilon = (\frac{1}{3})^n$ and $N(\epsilon) = 3.4^n$. Thus

$$d_c = \lim_{n\to\infty} (\log 3 + n \log 4)/(n \log 3) = \log 4/\log 3 = 1.2618\ldots \quad (7)$$

Here we have a dimension greater than 1 but less than 2. The fractal dimension can also be used to determine the dimension of an attractor. Other types of fractal dimensions include information dimension, correlation dimension, kth nearest-neighbor dimension, and Lyapunov dimension (14).

Lyapunov Exponents. The Lyapunov exponent λ can be used to measure the degree of dependence on initial conditions. It is easily computed for one-dimensional maps, such as the logistic map seen previously. Consider two different initial conditions, x and $x + \epsilon$. If λ is negative, the trajectories converge and there is no chaos. If λ is positive, then the trajectories are sensitive to initial conditions and chaos exists. For a one-dimensional map where $x_{n+1} = f(x_n)$,

$$f_n(x+\epsilon) - f_n(x) \approx \epsilon e^{\lambda n} \quad (8)$$

where $f_n(x)$ is the function at the nth iteration. Then

$$\ln[(f_n(x+\epsilon) - f_n(x))/\epsilon] \approx n\lambda \quad (9)$$

As ϵ approaches 0,

$$\lambda \approx (1/n) \ln |df_n|dx) \tag{10}$$

Using the chain rule,

$$\lambda = \lim_{n\to\infty} (1/n) \sum_{i=0}^{n-1} |f'(x_i)| \tag{11}$$

For n-dimensional maps, there are n Lyapunov exponents (9).

Central Tendency Measure. The central tendency measure quantifies the degree of variability in the second-order difference plots discussed previously (15). The CTM is computed by selecting a circular region around the origin of radius r, counting the number of points that fall within the radius, and dividing by the total number of points. Let t = total number of points, and r = radius of central area. Then

$$\text{CTM} = \left[\sum_{i=1}^{t-2} \delta(d_i)\right] \Big/ (t-2) \tag{12}$$

where

$$\delta(d_i) = \begin{cases} 1 & \text{if } [(a_{i+2} - a_{i+1})^2 + (a_{i+1} - a_i)^2]^{.5} < r \\ 0 & \text{otherwise} \end{cases}$$

The radius r is chosen depending on the character of the data (11).

EVALUATION OF EXPERIMENTAL DATA

Detection of Chaos

Many of the approaches discussed previously give measures for evaluating the presence of chaos of theoretical models. It becomes more difficult to verify the presence of data in experimental models. In the evaluation of time-series data collected experimentally, possible approaches for determination of the presence of chaos are the Lyapunov exponent, the fractal dimension, and the central tendency measure. The Lyapunov exponent and the fractal dimension are used in general to determine the presence or absence of chaos in the system. The central tendency measure is used differently in that it indicates the degree of chaos in the system.

Classification Using Chaotic Parameters

In many applications, especially in medicine, often the presence or absence of chaos in a time series such as the ECG is used to indicate the presence or absence of disease. Viewing the logistic equation for real values of n (i.e., for continuous rather than discrete solutions), it can be seen that the transition to chaos is not significant in the context of the time series. The degree of chaos is a better indicator for determination of presence of disease. Thus one of the numerical or graphical measures can be used directly for classification. In most cases, however, the problem is not that straightforward, and other parameters must be considered. If a numerical measure of chaos is used, it can then be combined with other parameters, such as results of laboratory tests and information from patient history and physical exam, in a decision-making model (such as neural network modeling, decision analysis, or Bayesian decision making) to determine the classification that indicates whether or not a particular disease is present.

Examples

Graphical Measures of Chaos in Disease Classification. Consider the case of ECG analysis. The ECG itself is a time series with a repeating pattern based on the QRS complex, with each pattern corresponding to a heartbeat. Thus it is not itself a chaotic time series. However, of more interest for the diagnosis of disease is the pattern of the R–R intervals, which is the time between heartbeats. Plotting the R–R intervals versus time gives a new time series that may or may not be chaotic. The second-order difference plot can be used effectively to demonstrate the character of the R–R intervals. Data is taken from 24 h Holter tape modeling, which produces in excess of 100,000 points. Figure 9(a) shows a second-order difference plot for a normal individual, while Fig. 9(b) shows a person with congestive heart failure (CHF) (13).

Numerical Measures of Chaos in Disease Classification. To obtain a numerical summary of information contained in the second-order difference plots, the central tendency measure (CTM) can be used. Central tendency measures that correspond to the plots in Fig. 9 are as follows:

$$\text{CTM}_\text{a} = 0.986 \quad \text{(total number of points =104,443)}$$
$$\text{CTM}_\text{b} = 0.232 \quad \text{(total number of points =109,374)}$$

where the CTM_a is the value for the normal case and CTM_b is the value for the diseased case. For these two cases, the CTM differs markedly for the normal individual and the patient with CHF. To investigate whether the CTM measure alone can be used as an indicator of congestive heart failure, 54 Holter tapes were analyzed: 26 for patients with CHF and 28 for normal subjects. Central tendencies were evaluated using $r = 0.1$ in Eq. (11). The analysis mean CTMs for CHF and normals of 0.69 and 0.90, respectively, are statistically significantly different at the $p = 0.01$ level. Only three normals had a CTM less than 0.8; 15 CHF subjects were in this category. No normal subjects were found with a CTM < 0.62 (14).

Chaotic Parameters in Decision Models. The CTM is one descriptor of the pattern seen in the second-order difference plots. It is possible to use more complex descriptions in a decision-making model through the use of neural network modeling (16). A nonstatistical neural network with a supervised learning algorithm was used to combine several measures. The result of the learning algorithm is a decision hypersurface:

$$D(\boldsymbol{x}) = \sum_{i=1}^{n} w_i x_i + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i \neq j}}^{n} w_{ij} x_{ij} \tag{13}$$

where $\boldsymbol{x}$ is an n-component vector with components (x_1, x_2, . . ., x_n), with each x_i representing an input parameter and each w_i indicating the relative weight associated with the pa-

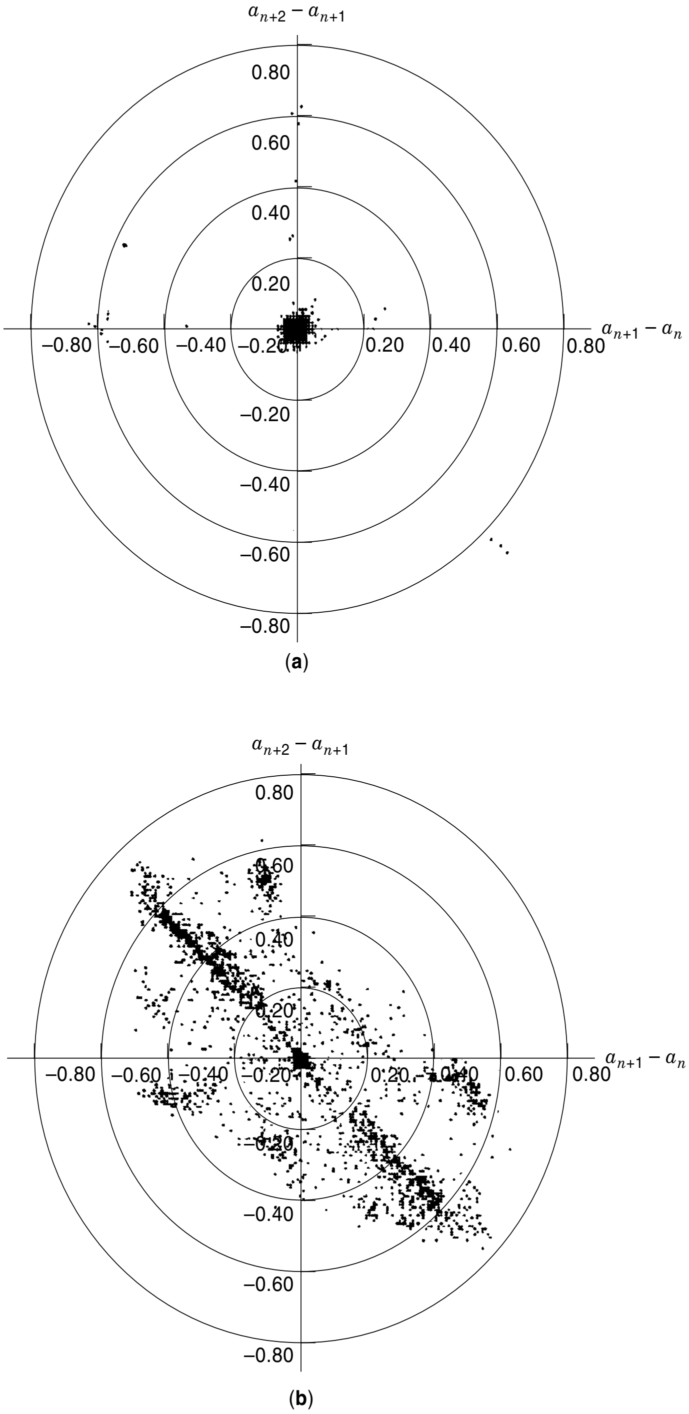

Figure 9. Second-order difference plots for normal individual (low variability) (a) and patient with CHF (high variability) (b).

rameter in the decision process. The classification is then made according to

$$\begin{array}{ll} D(\boldsymbol{x}) > 0 & \text{Case belongs to class 1} \\ D(\boldsymbol{x}) < 0 & \text{Case belongs to class 2} \\ D(\boldsymbol{x}) = 0 & \text{Indeterminate} \end{array}$$

The neural network was used to establish a decision with four values: CTM with $r = 0.05$, CTM with $r = 0.10$, total number of R–R intervals in 24 h, and lowest value of r for which CTM > 0.99. The model resulted in sensitivity of 80%, specificity of 89%, and overall accuracy of 85%. Using this approach, it is also possible to combine measures of chaos with clinical parameters in the same decision models. In another study, clinical parameters were recorded for 32 patients with CHF and 20 patients with other types of cardiovascular disease. Holter tapes were analyzed using the CTM method. Using the CTM measure alone, if a cutoff value of 0.87 is used, only one non-CHF patient has a CTM value below the cutoff, while 22 of the 32 CHF patients have CTM values below the cutoff, showing that this is indeed a strong parameter in the decision process. For the model that includes clinical parameters, the neural network selected the following clinical parameters: edema, rales, heart rate, concentration of nitrogen in the form of urea in the blood (BUN), CTM ($r = 0.1$). The first three of these are physical findings and the fourth is a test result. Using these five parameters, the model was able to classify cases with a sensitivity of 84%, a specificity of 82%, and an accuracy of 84% (16).

The CTM as it is used in these examples gives a measure of the degree of chaos that is seen to be useful in the analysis of experimental data.

CONTINUOUS CHAOTIC MODELING VERSUS DISCRETE CHAOTIC MODELING

Most approaches to chaotic modeling involve discrete models, although the processes that they represent are continuous. The process of discretization may, in fact, introduce problems, including singularities. Chaos may occur only in the strict mathematical sense. These problems can be illustrated by looking at the continuous solution of Eq. (1). As mentioned previously, there are exact solutions of Eq. (1) for $A = 2$ and $A = 4$. The solution at $A = 4$ is of particular interest since it falls within the region of chaos. The solution is

$$a_n = \tfrac{1}{2}[1 - T_{2n}(1 - 2a_0)] \tag{14}$$

where $T_n(x)$ is the Chebyshev function, valid for all real values of n.

This solution has a number of interesting properties that emphasize that it is indeed a well-behaved function. It is indeed orthogonal (17), satisfying the relation

$$\int_0^1 f_n(a_0) f_m(a_0)[a_0(1-a_0)]^{-1/2} da_0 = \begin{array}{ll} 0 & n \neq m \\ B & n = m \end{array} \tag{15}$$

where

$$f_n(a_0) = \frac{(a_{n+1})^{\ell}}{1 - 2a_{n-1}} \quad \text{for} \quad \begin{array}{l} \ell = 1, 2, 3, \ldots \\ n = 1, 2, 3, \ldots \\ m = 1, 2, 3, \ldots \end{array}$$

$$B = \frac{(\ell\pi)4^{\ell+1}(1/2)_{2\ell}(1/2)_{2\ell-1}}{(4\ell)!}$$

where

$$\begin{array}{ll} (c)_k = c(c+1)\ldots(c+k-1), & k \geq 1 \\ \quad\;\; = 1 & k = 0 \end{array}$$

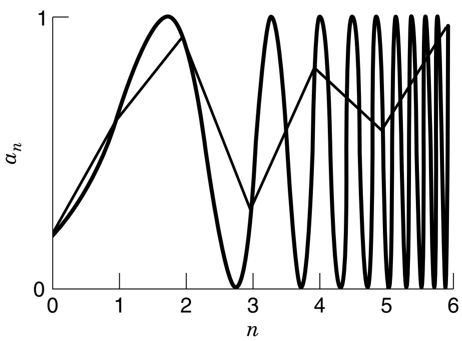

Figure 10. Exact solution of logistic equation at $A = 4$, $a_0 = 0.2$ compared to discrete values of n in Eq. (1), demonstrating well-behaved nature of continuous solution.

Note that B reduces to π for $\ell = 1$, an interesting special case.

Figure 10, for $A = 4$, shows a graph with the integer values generated by Eq. (1) (curved line) along with the continuous solution (straight line) for n between 0 and 6 inclusive, with initial value $a_0 = 0.2$. Note that while examination of integer values only results in a graph with an apparently arbitrary pattern, the continuous solution shows a well-behaved oscillatory function. Figure 11 shows a similar graph with an initial value of $a_0 = 0.5$. If only integer values are considered, it appears that Eq. (1) converges to zero for this initial value. The exact solution demonstrates that this is not the case.

The approximation given in Eq. (2) is valid only for $A = 4$, a point in the region of chaos. As no exact solution is available for other values within the region of chaos, we constructed a method for approximating solutions for any value of A, $2 \leq A \leq 4$.

Assume a solution of the type

$$a_n = \sum_{k=0}^{\ell} \alpha_k T_k(2^n x) \qquad (16)$$

where $T_k(x)$ is the Chebyshev function of the first kind and n is a real number. We assume ℓ to be the number of points in the interval $0 \leq n \leq 1$. Thus

$$a_n^2 = \sum_{k=0}^{\ell} \alpha_k^2 T_k^2(2^n x) + 2 \sum_{\substack{j>i \\ i=0,1,\ldots,\ell-1 \\ j=1,2,\ldots,\ell}} \alpha_i \alpha_j T_i(2^n x) T_j(2^n x) \qquad (17)$$

The conjecture adopted is that going from one point to another implies adding a Chebyshev polynomial. Hence

$$a_{n+1} = \sum_{k=0}^{2\ell} \beta_k T_k(2^n x) \qquad (18)$$

where n is assumed to be a real number.

Feeding Eqs. (16), (17), and (18) into the logistic equation [Eq. (1)], simplifying, and comparing coefficients gives nonlinear equations involving the unknowns α_i's and β_j's and the arguments x of the Chebyshev polynomials. Solving 300 equations involving 300 variables and choosing the appropriate α_i's and β_j's and the argument of the Chebyshev to satisfy a_n to be strictly monotonic increasing in the interval $0 \leq n \leq 1$ produces the approximate solution. It should be pointed out that the nonlinear equations give a multitude of solutions. By imposing appropriate boundary conditions, one obtains a unique solution to these nonlinear equations involving 300 variables. Values for $n > 1$ are obtained by applying the logistic equation to the points obtained for $0 \leq n \leq 1$.

It is important to note that the approximate solution permits the extension of the solution to be valid not only for integers but for all real values of n, and for all values of A in the range of interest, $2 \leq A \leq 4$. Figure 12 shows the conjectured solution (straight line) superimposed on the exact solution (curved line) for $A = 4$. Figure 13 shows a graph using the conjectured solution at $A = 3.55$ and $A = 3.6$, values on each side of the onset of chaos. Note that there is no significant change when the continuous solution is considered. Also note that this plot is consistent with the bifurcations found in the discrete solution.

The concept of continuous chaotic modeling raises a number of issues in the analysis of chaotic systems (18). Many discrete approximations to continuous models have limita-

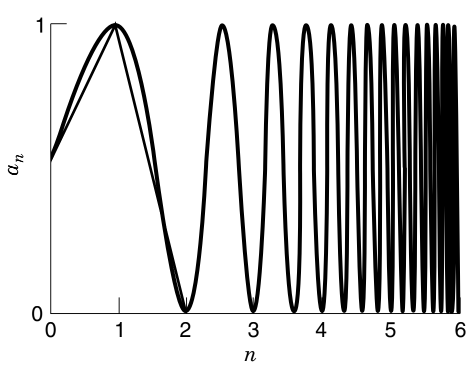

Figure 11. Exact solution of logistic equation at $A = 4$, $a_0 = 0.5$ compared to discrete values of n in Eq. (1) illustrating misleading perception of discrete view.

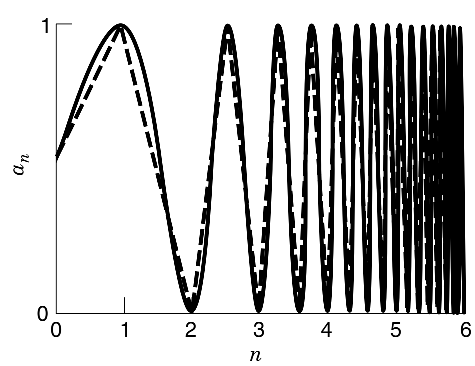

Figure 12. Comparison of conjectured solution of logistic equation to exact solution at $A = 4.0$ showing close agreement.

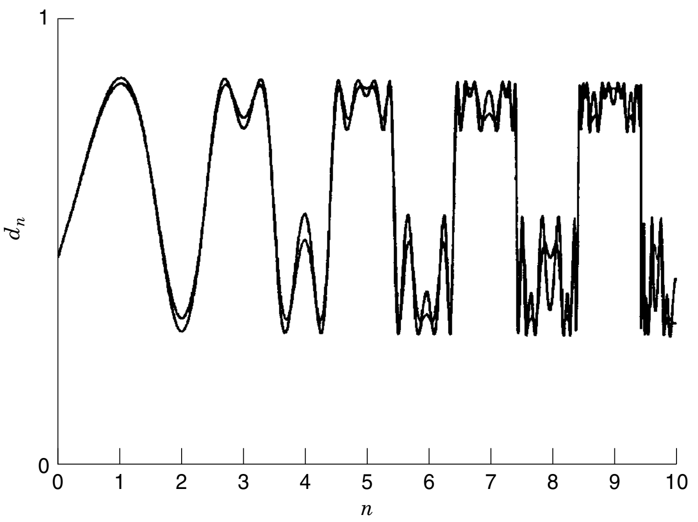

Figure 13. Conjectured solution at $A = 3.55$ and $A = 3.57$ showing no abrupt change from nonchaotic to chaotic state.

tions that must be observed. If boundary conditions are not known, it is difficult to verify if the discrete solutions correspond to the original model. This proviso applies not only to the discrete models of chaotic equations but to other discrete mathematical approaches, such as numerical solution of the partial differential equations through methods such as the finite element approach, as well as to the use of the fast Fourier transform, the discrete analog of the Fourier transform. This is not to say that these methods should not be utilized, but that solutions must be verified to correspond to known boundary conditions. In chaotic systems, this is often difficult. Due to the sensitivity to initial conditions, round-off error caused by computer representation can come into play. Even with very long word length in double precision using the Cray computer, errors will occur after a few hundred iterations. Under some conditions, these errors can overtake the actual solution.

Chaos theory is a powerful tool in the analysis of nonlinear problems, particularly in its application to time series.

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CHAOTIC CARRIER SIGNALS.

See TRANSMISSION USING CHAOTIC SYSTEMS.