FREQUENCY STABILITY

Stable and spectrally pure signal generators have been widely employed in various fields of science and technology such as physics, high precision frequency standards, fundamental metrology, telecommunication systems, space missions, radars, and broadcasting. In addition to the inherent fluctuations in the output signal, almost all signal generators are influenced to some extent by their environment, such as changes in ambient temperature, supply voltage, magnetic field, barometric pressure, humidity, and mechanical vibration. These perturbations manifest themselves as noise in frequency or phase of the output signals and become the limiting factor in various applications. Therefore, it is of fundamental importance to characterize the frequency fluctuations in the output signal using a common measure as well as to reduce these fluctuations to an acceptable level.

Extensive research efforts have been devoted to the establishment of a general and common measure of frequency stability in the past 50 years. Tremendous progress has been achieved since 1955 through the development of high precision frequency standards, such as cesium and rubidium clocks, hydrogen masers, and quartz crystal oscillators. In the early 1960s the needs were clearly recognized for common parameters characterizing the frequency stability and for related measurement techniques. These parameters were required for at least two main purposes: The first purpose is to allow for meaningful comparisons between similar devices developed by different laboratories or between different devices in a given application; and the second one is to access application performance in terms of the measured oscillator frequency stability.

In 1964 a special symposium on frequency stability was organized by the National Aeronautics and Space Administration (*NASA*) and the Institute of Electrical and Electronics Engineers (*IEEE*) as an attempt to improve the situation (1). After this symposium, a Subcommittee on Frequency Stability was formed as a part of the Technical Committee on Frequency and Time of the IEEE Professional Group on Instrumentation and Measurement. In 1966 several members of this subcommittee contributed an original paper to a special issue of the *Proceedings of the IEEE* (2). In 1970 a report on the characterization of frequency stability was issued by the subcommittee mentioned above (3). In 1988, IEEE updated the standard on frequency stability and published IEEE Std 1139-1988, Standard Definitions of Physical Quantities for Fundamental Frequency and Time Metrology. In 1999 this standard was revised and published as IEEE Std 1139-1999, Standard Definitions of Physical Quantities for Fundamental Frequency and Time Metrology — Random Instabilities. The recommended measures of instabilities in frequency generators have been widely accepted among frequency and time users throughout the world.

In this article, definition and estimation procedure will be presented on the measure of frequency stability commonly employed. For more extensive and complete discussion, refer the IEEE Standards, text book and reviews on this subject (4, 5).

BACKGROUND AND DEFINITION

Consider a sinusoidal signal generator whose instantaneous output voltage $u(t)$ may be written as

$$u(t) = [U_0 + \epsilon(t)]\sin[2\pi\nu_0 t + \phi(t)] \tag{1}$$

where U_0 and n_0 are the nominal values of the amplitude and frequency, respectively. Throughout this article the Greek letter ν is used to stand for the signal frequency, while the Latin symbol f is used to denote the Fourier frequency in the representation of spectral densities. The parameter $\epsilon(t)$ and $\phi(t)$ in Eq. (1) represent the deviations from the nominal amplitude and phase, respectively. The instantaneous frequency $\nu(t)$ of the sinusoidal voltage is then expressed as the sum of a constant value ν_0 and variable term $\nu_v(t)$:

$$\nu(t) = \nu_0 + \frac{1}{2\pi}\frac{d\phi(t)}{dt} = \nu_0 + \nu_\upsilon(t) \tag{2}$$

Since we are dealing with stable oscillators, it is assumed that the magnitude of these fluctuations are much smaller than the nominal values; that is,

$$|\epsilon(t)| \ll U_0 \tag{3}$$

and

$$|\nu_\upsilon(t)| \ll \nu_0 \tag{4}$$

for substantially all time t. Oscillators with large frequency deviation are a subject of frequency modulation theory which is not treated in this article. Various types of oscillators are used in scientific and engineering fields, and their nominal frequencies cover a wide range of the spectrum—that is, from several hertz to several hundred terahertz (lightwave). For a general discussion of oscillators having a wide range of frequencies, it is useful to introduce normalization. Frequency instability of an oscillator is defined in terms of the instantaneous, normalized frequency deviation, $y(t)$, as follows:

$$y(t) = \frac{\nu_\upsilon(t)}{\nu_0} = \frac{1}{2\pi\nu_0}\frac{d\phi(t)}{dt} \tag{5}$$

Time deviation, defined in terms of phase deviation $\phi(t)$, is expressed by the time integral of $y(t)$:

$$x(t) = \int_0^t y(t)\,dt = \frac{\phi(t)}{2\pi\nu_0} \tag{6}$$

and

$$y(t) = \frac{dx(t)}{dt} \tag{7}$$

The parameter $x(t)$ has the dimension of time and is proportional to instantaneous phase. This quantity was originally named as "phase-time" (Phasenzeit) by Becker (6).

From the definition of $y(t)$ given by Eq. (5), it is natural to assume that $y(t)$ should have a zero mean over the time of observation, whereas this will not be the case for $x(t)$. However, most actual oscillators exhibit frequency drift with time as well as random variations. In performing a series of measurements over a long period of time, it is always possible to subtract the drift and the initial offset from the data.

There are two aspects in the analysis of measured results of $y(t)$, namely, time-domain analysis and frequency-domain analysis.

FREQUENCY DOMAIN

The behavior of frequency deviation $y(t)$ in the frequency domain is described by its power spectral density $S_y(f)$, which is defined as the Fourier transform of the autocorrelation function $R_y(\tau)$ given by

$$R_y(\tau) = \lim_{T\to\infty} \int_0^T y(t')y(t'+\tau)\,dt' \tag{8}$$

From the Wiener–Khintchine theorem, the power spectral density is obtained from the autocorrelation function as

$$S_y(f) = 4\int_0^\infty R_y(\tau)\cos 2\pi f\tau\, d\tau \tag{9}$$

and inversely

$$R_y(\tau) = \int_0^\infty S_y(f)\cos 2\pi f\tau\, df \tag{10}$$

The power spectral density represents the fluctuation power of the frequency deviation $y(t)$ contained in a unit bandwidth as a function of Fourier frequency. Its dimension is Hz^{-1}, since $y(t)$ and $R_y(\tau)$ are dimensionless.

The relations between the power spectral densities of various quantities are shown below.

Absolute frequency deviation: $\delta\nu = \nu(t) - \nu_0$[Hz]

$$S_{\delta\nu}(f) = \nu_0^2 S_y(f)\ [\mathrm{Hz}] \tag{11}$$

Phase: $\phi(t)$ [radian]

$$S_\phi(f) = \frac{\nu_0^2}{f^2} S_y(f)\ [(\mathrm{radian})^2/\mathrm{Hz}] \tag{12}$$

Time: $x(t) = \int_0^t y(t)\,dt$ [s]

$$S_x(f) = \frac{1}{(2\pi f)^2} S_y(f)\ [(\mathrm{radian})^{-2}/\mathrm{Hz}] \tag{13}$$

Angular frequency: $\omega = {}^2\phi = d\phi/dt$ [radian/s]

$$S_\omega(f) = (2\pi\nu_0)^2 S_y(f)\ [(\mathrm{radian})^2\ \mathrm{Hz}] \tag{14}$$

The above relations are very useful for converting measured data and for translating formula between various measures.

TIME DOMAIN

The measure of frequency stability in the time domain is based on the sample variance of the fractional frequency deviation. In actual measurements it is difficult to obtain the instantaneous sample of the frequency deviation $y(t)$. The results of frequency measurement are always in the form of sampled data of $\bar{y}_k$ of $y(t)$ averaged over a finite time interval τ, which is given by

$$\bar{y}_k(t_k, \tau) \equiv \frac{1}{\tau}\int_{t_k}^{t_k+\tau} y(t)\,dt = \frac{\phi(t_k+\tau) - \phi(y_k)}{2\pi\nu_0\tau} \tag{15}$$

where $t_{k+1} = t_k + T, k = 0, 1, 2, \ldots, T$ represents the repetition interval for measurements of duration τ, and t_0 is arbitrary. The dead time between measurements is given by $T - \tau$. Figure 1 shows the measurement process for the sampled data $\bar{y}_k$.

The measure of frequency stability in the time domain is then defined in analogy to the sample variance by the relation

$$\sigma_y^2(N, T, \tau) \equiv \left\langle \frac{1}{N-1}\sum_{n=1}^{N}\left(\bar{y}_n - \frac{1}{N}\sum_{k=1}^{N}\bar{y}_k\right)^2 \right\rangle \tag{16}$$

where $\langle\rangle$ denotes the infinite time average. The quantity in Eq. (16) is called the *Allan variance* (7) and is dimensionless. The square root of the Allan variance $\sigma_y(N,T,\tau)$ is called the Allan deviation.

In many situations it is not correct to assume that the variance (16) converges to a meaningful limit as $N \to \infty$. In practice one cannot let N approach infinity, and it is known that some actual noise processes contain substantial fractions of the total noise power in the extremely low Fourier frequency range. Therefore it is important to specify particular N and T in order to improve the comparability of data.

The recommended definition of frequency stability is choosing $N = 2$ and $T = \tau$, which means no dead time between measurements. Expressing $\langle\sigma^2{}_y(N = 2, T = \tau, \tau)\rangle$ as $\sigma^2{}_y(\tau)$, the Allan variance may be written as

$$\sigma_y^2(\tau) = \left\langle \frac{(\bar{y}_{k+1} - \bar{y}_k)^2}{2} \right\rangle \tag{17}$$

In a practical situation, the true time average is not realizable and the estimate of $\sigma^2{}_y(\tau)$ should be obtained from a finite number of samples. It has been shown that when $\sigma^2{}_y(\tau)$ is estimated from the ensemble average of $\sigma^2{}_y(2, \tau, \tau)$, the average converges with increasing the number of data even for noise processes that do not converge for $\langle\sigma^2{}_y(N, \tau, \tau)\rangle$ as $N \to \infty$. Therefore, $\sigma^2{}_y(\tau)$ has greater utility than $\langle\sigma^2{}_y(\infty, \tau, \tau)\rangle$ even though both involve an assumption of an infinite average. A widely used formula for estimating $\sigma^2{}_y(\tau)$ experimentally is

$$\sigma_y^2(\tau, m) = \frac{1}{2(m-1)}\sum_{k=1}^{m-1}(\bar{y}_{k+1} - \bar{y}_k)^2 \tag{18}$$

where m represents the number of samples. Another advantage of the Allan variance with $N = 2$ and $T = \tau$ is the simplicity of computation from the measured data as shown by Eq. (18). In any case one should specify the number of samples used for estimation in order to avoid ambiguity and to allow for meaningful comparisons.

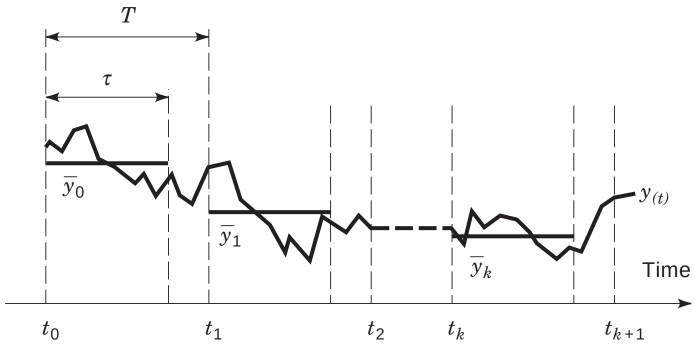

Figure 1. Measurement process for the calculation of sample variances. Here T and τ represent the repetition interval and duration of the measurement, respectively.

TRANSLATIONS BETWEEN MEASURES (FREQUENCY DOMAIN TO TIME DOMAIN)

The relation between the Allan variance and the power spectral density $S_y(f)$ was derived by Cutler and co-workers (3) and is expressed as

$$\langle\sigma_y^2(N,T,\tau)\rangle = \frac{N}{N-1}\int_0^\infty S_y(f)|H(f)|^2\,df \tag{19}$$

where

$$|H(f)|^2 = \frac{\sin^2 \pi f\tau}{(\pi f\tau)^2}\left(1-\frac{\sin^2 \pi r f\tau N}{N^2\sin^2 \pi r f\tau}\right) \tag{20}$$

and

$$r = T/\tau \tag{21}$$

The parameter r represents the ratio of the time interval between successive measurements to the duration of the averaging periods. Equation (19) allows one to estimate the time-domain stability $\langle\sigma^2{}_y(N,T,\tau)\rangle$ from the frequency domain stability $S_y(f)$. Two assumptions were made in deriving Eq. (19). The first one is that $R_y(t_1 - t_2) = \langle y(t_1)y(t_2)\rangle$ exists; that is, $y(t)$ is stationary in the covariance sense. The second assumption is that

$$\langle y^2(t)\rangle = R_y(0) = \int_0^\infty S_y(f)\,df \tag{22}$$

exists. To satisfy the second assumption it is sufficient to assume that $S_y(f)$ is finite in the frequency interval $f_l < f < f_h$ and zero outside this interval; that is, there are lower and higher cutoff frequencies in the device and the measuring equipment. This condition is always satisfied in practical situations.

In the limit of $N \to \infty$, Eq. (19) reduces to

$$\lim_{N\to\infty}\langle\sigma_y^2(N,T,\tau)\rangle = \frac{1}{\pi\tau}\int_0^\infty S_y\left(\frac{u}{\pi\tau}\right)\frac{\sin^2 u}{u^2}\,du \tag{23}$$

$$u = \pi f\tau \tag{24}$$

In the special case of $N = 2$, Eq. (19) is written by

$$\langle\sigma_y^2(2,T,\tau)\rangle = \frac{2}{\pi\tau}\int_0^\infty S_y\left(\frac{u}{\pi\tau}\right)\frac{\sin^2 u\sin^2 ru}{u^2}\,du \tag{25}$$

By comparing Eqs. (23) and (24), it can be seen that the convergence on the lower limit is better for $N = 2$ because of the additional factor of $\sin^2 ru$. For the Allan variance with $N = 2$ and $T = \tau$, the translation is expressed as

$$\sigma_y^2(\tau) = 2\int_0^\infty S_y(f)\frac{\sin^4 \pi f\tau}{(\pi f\tau)^2}\,df \tag{26}$$

TRANSLATIONS BETWEEN MEASURES (TIME DOMAIN TO FREQUENCY DOMAIN)

For general $\langle\sigma^2{}_y(N,T,\tau)\rangle$ no simple prescription is available for translation into the frequency domain. For this reason, $S_y(f)$ is preferred as a general measure of frequency stability, especially for theoretical work. For specific types of noise process discussed below, the Allan variance $\sigma^2{}_y(\tau)$ can be translated into the power spectral density $S_y(f)$.

OSCILLATOR NOISE MODEL

The types of noise observed on the output signal of actual oscillators can be suitably represented by the spectral density $S_y(f)$. It has been known empirically that a simple power-law model of the form

$$S_y(f) = h_{-2}f^{-2} + h_{-1}f^{-1} + h_0 + h_1 f + h_2 f^2 = \sum_{\alpha=-2}^{2} h_\alpha f^\alpha$$
$$S_y(f) = 0 \qquad \text{for } f > f_h \tag{27}$$

can cover all actually known types of oscillators in the limit of drift elimination. In the above equation, h_α ($\alpha = -2, -1, 0, 1, 2$) is a constant. It is assumed that the measuring system has an ideally sharp upper cutoff frequency f_h. The individual terms have been identified by common names given in Table 1. Figure 2 shows the power spectral density $S_y(f)$ for five noise processes in Eq. (27). It can be seen that each noise process is clearly distinguishable from the slope.

It is easy to show the relationship between $S_y(f)$ defined above and $\sigma^2{}_y(\tau)$ by using the translation of Eq. (26). For every term of the form $h_\alpha f^\alpha$($\alpha = -2, -1, 0, 1, 2$) we have

$$\sigma_y^2(\tau) = \frac{2h_\alpha}{(\pi\tau)^{\alpha+1}}\int_0^{\pi\tau f_h} u^{\alpha-2}\sin^4 u\,du \tag{28}$$

where $u = \pi f\tau$. This relation may be expressed as

$$\sigma_y^2(\tau) = K_\alpha\tau^\mu \tag{29}$$

$$\mu = -\alpha - 1 \tag{30}$$

Table 1. Common Names of Various Noise Terms

Random walk y	$\alpha = -2$	$\mu = 1$
Flicker y	$\alpha = -1$	$\mu = 0$
White y		
Random walk x	$\alpha = 0$	$\mu = -1$
Flicker x	$\alpha = 1$	$\mu = -2$
White x	$\alpha = 2$	$\mu = -2$

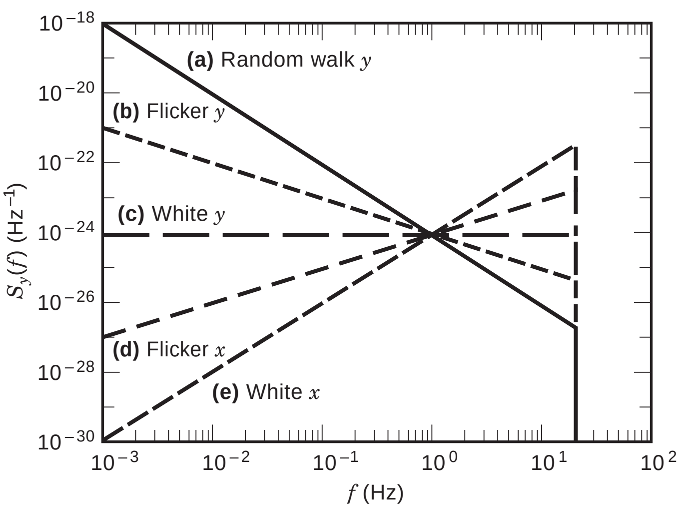

Figure 2. Power spectral density $S_y(f)$ for five noise terms in Eq. (27). A sharp upper cutoff frequency of f_h = 20 Hz is assumed. (a) Random walk frequency noise, (b) flicker frequency noise, (c) white frequency noise, (d) flicker phase noise, (e) white phase noise. $h_{-2} = h_{-1} = h_0 = h_1 = h_2 = 10^{-24}$.

$$K_\alpha = \frac{2h_\alpha}{(\pi\tau)^{\alpha+1}} \int_0^{\pi\tau f_h} u^{\alpha-2} \sin^4 u \, du \tag{31}$$

For $\alpha < 1$ and $2\pi\tau f_h > 1$, K_α is independent of f_h and τ and becomes a constant due to the very rapid convergence of the integral. For $\alpha = 1$ and 2, the value of integral depends on f_h as well as on τ. The relations for general N and r are shown in Table 2. Figure 3 shows the dependence of $\sigma_y(\tau)$ on averaging time τ for five noise processes in the limit of $2\pi\tau f_h > 1$. It can be seen that the noise processes with α = 0, −1, −2 are clearly distinguished from the slope of $\sigma_y(\tau)$. However, the slope of $\sigma_y(\tau)$ is almost the same for $\alpha = 1$ and $\alpha = 2$. As a consequence, the Allan variance $\sigma_y(\tau)$ is not useful for distinguishing flicker and white noise processes. For both types of noise, the dominant contribution to $\sigma_y(\tau)$ is frequency fluctuations at $f = f_h$ even for long measurement time. Therefore the determination of $\sigma_y(\tau)$ for some types of noise is dependent on the noise bandwidth and on the type of low-pass filter used in the measurement.

The power expansion law of Eq. (27) has some physical meaning. Any practical oscillator contains a frequency-determining element (resonant circuit, quartz crystal resonator, atomic resonator, optical resonator) and a feedback loop. Any sources of noise have influences on the frequency or on the phase of the generated signal. Therefore, it is also useful to treat the noise in terms of phase fluctuations. Using Eq. (13) to transform Eq. (27) we can define the power spectral density of the time deviation

$$\begin{aligned} S_x(f) &= \frac{1}{4\pi^2} \sum_{\alpha=-2}^{2} h_\alpha f^{\alpha-2} \\ &= \frac{1}{4\pi^2}(h_{-2}f^{-4} + h_{-1}f^{-3} + h_0 f^{-2} + h_1 f^{-1} + h_2) \end{aligned} \tag{32}$$

The various types of noise characterized by α terms in Eqs. (27) and (32) usually dominate over some frequency range; in most cases, not all five terms are significant. The individual terms have been identified by common names given in Table 1.

The types of noise commonly observed in oscillators are as follows:

1. Additive Noise Thermal noise in amplifier is simply added to the signal. This type of noise appears as phase noise and is usually white with high cutoff frequency.
2. Perturbing Noise Thermal and shot noise acting within the feedback loop appears as white frequency noise (random walk in phase).
3. Modulating Noise This type of noise is caused by random variations of reactive parameters such as semiconductor junction capacitors, frequency-determining parameters of resonators, and so on. These fluctuations are either inherent in the devices or due to environmental effect. Flicker x and y usually belong to this class.

Table 2. Stability Measure Conversion Chart

Frequency Domain (Power Spectral Densities)	Time Domain (Allan Variances) $\sigma_y^2(\tau)$ $[N = 2, r = 1]$	$\langle \sigma_y^2(N, T = \tau, \tau, f_h) \rangle$ $[r = 1]$	$\{\sigma_y^2(N, T, \tau, f_h)\rangle$
White x $S_y(f) = h_2 f^2$ $S_x(f) = h_2/(2\pi)^2$ $2\pi f_h \tau \gg 1$	$h_2 \cdot \frac{3 f_h}{(2\pi)^2 \tau^2}$	$h_2 \cdot \frac{N+1}{N(2\pi)^2} \cdot \frac{2 f_h}{\tau^2}$	$h_2 \cdot \frac{N + \delta_k(r-1)}{N(2\pi)^2} \cdot \frac{2 f_h}{\tau^2}$ $\delta_k(r-1) = 1, r = 1$ $= 0, otherwise$
Flicker x $S_y(f) = \mathrm{h}_1 f$ $S_x(f) = h_1/(2\pi)^2 f$ $2\pi f_h \tau \gg 1, 2\pi f_h T \gg 1$	$h_1 \cdot \frac{\frac{9}{2} + 3\ln(2\pi f_h \tau) - \ln 2}{(2\pi)^2 \tau^2}$	$h_1 \cdot \frac{2(N+1)}{N\tau^2(2\pi)^2}\left[\frac{3}{2} + \ln(2\pi f_h \tau) - \frac{\ln N}{N^2 - 1}\right]$	$h_1 \cdot \frac{2}{(2\pi\tau)^2}\left\{\frac{3}{2} + \ln(2\pi f_h \tau) + \frac{1}{N(N-1)} \times \sum_{n=1}^{N-1} (N-n) \cdot \ln\left[\frac{n^2 r^2}{n^2 r^2 - 1}\right]\right\}, r \gg 1$
White y (random walk x) $S_y(f) = h_0$ $S_x(f) = h_0/(2\pi)_2 f^2$	$h_0 \cdot \frac{1}{2}\tau^{-1}$	$h_0 \cdot \frac{1}{2}\tau^{-1}$	$h_0 \cdot \frac{1}{2}\tau^{-1}, r \geq 1$ $h_0 \cdot \frac{1}{6} r(N+1)\tau^{-1}, Nr \leq 1$
Flicker y $S_y(f) = \frac{h_{-1}}{f}$ $S_x(f) = h_{-1}/(2\pi)^2 f^3$	$h_{-1} \cdot 2 \ln 2$	$h_{-1} \cdot \frac{N \ln N}{N-1}$	$h_{-1} \cdot \frac{1}{N(N-1)} \sum_{n=1}^{N-1} (n-n)\,[-2(nr)^2 \ln(nr) + (nr+1)^2 \ln(nr+1) + (nr-1)^2 \ln\lvert nr-1 \rvert]$
Random walk y $S_y(f) = \frac{h_{-2}}{f^2}$ $S_x(f) = h_{-2}/(2\pi)^2 f^4$	$h_{-2} \cdot \frac{(2\pi)^2 \tau}{6}$	$h_{-2} \cdot \frac{(2\pi)^2 \tau}{12} \cdot N$	$h_{-2} \cdot \frac{(2\pi)^2 \tau}{12}[r(N+1) - 1], r \geq 1$

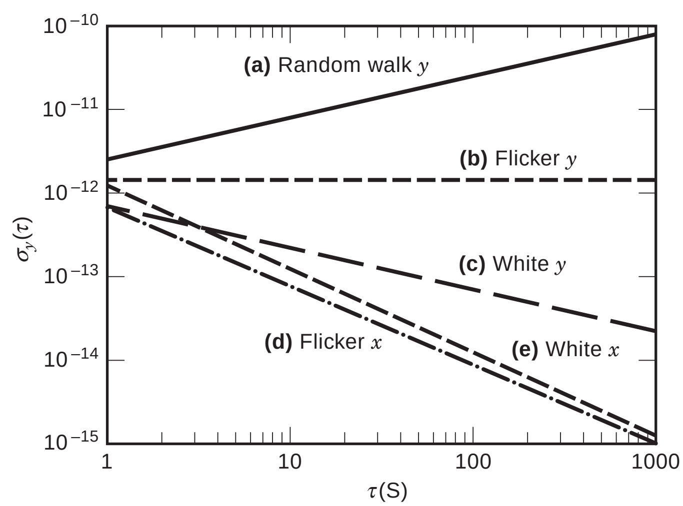

Figure 3. Square root of the Allan variance $\sigma_y(\tau)$ for five noise processes in Eq. (27) in the limit of $2\pi\tau f_h > 1$, where f_h represents the sharp upper cutoff frequency. (a) Random walk frequency noise $S_y(f) = h_{-2}f^{-2}$, (b) flicker frequency noise $S_y(f) = h_{-1}f^{-1}$, (c) white frequency noise $S_y(f) = h_0$, (d) flicker phase noise $S_y(f) = h_1 f^{-}$ with f_h = 20 Hz, (e) white phase noise $S_y(f) = h_2 f^{2-}$ with f_h = 20 Hz. $h_{-2} = h_{-1} = h_0 = h_1 = h_2 = 10^{-24}$.

MODIFIED ALLAN VARIANCE

To improve the relatively poor discrimination of the Allan variance $\sigma_y(\tau)$ against flicker ($\alpha = 1$) and flicker ($\alpha = 2$) phase noise, the modified Allan variance was introduced in 1981. The definition is based on (a) the algorithm developed by Snyder (8) for increasing the resolution of frequency meters and (b) the detailed consideration of Allan and Barnes (9).

It consists in dividing a time interval τ into n cycles of period τ_0 such that

$$\tau = n\tau_0. \tag{33}$$

As depicted in Fig. 4, for a given observation time interval of duration 2τ, there are n overlapping time intervals of duration τ. Allan and Barnes introduced the modified Allan variance such that

$$\mathrm{Mod}\,\sigma_y^2(\tau) = \frac{1}{2\tau^2} \left\langle \left[\frac{1}{n} \sum_{k=1}^{n} \left\{ \int_{t_0+(k+n)\tau_0}^{t_0+(k+2n)\tau_0} y(t)\,dt - \int_{t_0+k\tau_0}^{t_0+(k+n)\tau_0} y(t)\,dt \right\} \right]^2 \right\rangle \tag{34}$$

It can be seen from the above equation that the calculation of each statistical sample involved in the definition of Mod $\sigma^2{}_y(\tau)$ requires a signal observation of duration 3τ.

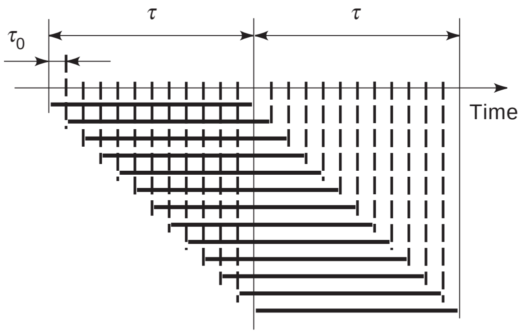

Figure 4. Measurement process for the calculation of the modified Allan variance.

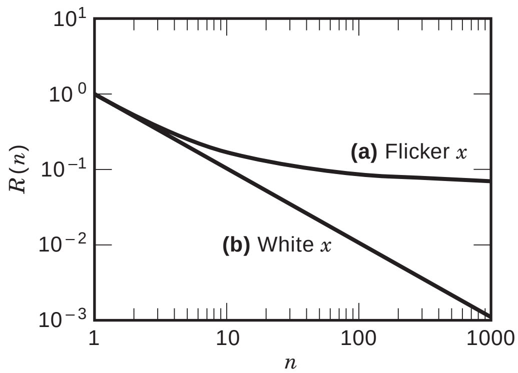

Figure 5. Dependence of the ratio $R(n)$ on n for (a) flicker and (b) white phase noise processes under the condition $2\pi\tau f_h > 1$.

For $n = 1$, the Allan variance and the modified one are equal.

$$\text{Mod}\,\sigma_y^2(\tau) = \sigma_y^2(\tau) \tag{35}$$

The power spectral density $S_y(f)$ is translated into the modified Allan variance Mod $\sigma^2{}_y(\tau)$ by the following relation:

$$\text{Mod}\,\sigma_y^2(\tau) = \frac{2}{n^2\pi^2\tau^2}\left[n\int_0^\infty \frac{S_y(f)}{f^2}\sin^4(\pi f n\tau_0)\,df + 2\sum_{k=1}^{n-1}(n-k)\int_0^\infty \frac{S_y(f)}{f^2} \cos(2\pi f k\tau_0)\sin^4(\pi f n\tau_0)\,df\right] \tag{36}$$

The analytical expressions of the modified Allan variance for each noise term in Eq. (27) can be directly calculated by the above equation. These relations are summarized in Table 3 in the limit of $2\pi\tau f_h > 1$ (10).

It is useful for comparing the Allan variance with the modified one to define the ratio $R(n)$ as

$$R(n) = \frac{\text{Mod}\,\sigma_y^2(\tau)}{\sigma_y^2(\tau)} \tag{37}$$

Figure 5 depicts the variation of the ratio $R(n)$ with n, for flicker and white phase noise processes. It can be seen that for a large value of n, flicker and white phase noise processes have different dependencies. This property can be used to distinguish these two types of noise processes.

For application to time transfer systems, such as the global positioning system (GPS), $\sigma_x^2(\tau)$ is often used, which is defined as

$$\sigma_x^2(\tau) = \frac{\tau^2}{3}\text{Mod}\,\sigma_y^2(\tau) \tag{38}$$

This measure is useful when flicker and white phase noise are domunant.

EXAMPLE OF THE ALLAN VARIANCE

Sinusoidal Frequency Modulation

Consider a frequency-modulated (*FM*) signal with the modulation frequency of f_m and maximum frequency deviation of $\Delta\nu_0$. The instantaneous, normalized frequency deviation $y(t)$ is given by

$$y(t) = \frac{\Delta\nu_0}{\nu_0}\sin(2\pi\ f_m t). \tag{39}$$

The power spectral density for this signal is expressed as

$$S_y(\ f) = \frac{1}{2}(\frac{\Delta\nu_0}{\nu_0})^2\delta(\ f -\ f_m), \tag{40}$$

where δ is the Dirac delta function. Substitution of Eq. (40) into Eq. (26) and (36) yields

$$\sigma_y^2(\tau) = (\frac{\Delta\nu_0}{\nu_0})^2\frac{\sin^4(\pi\ f_m\tau)}{(\pi\ f_m\tau)^2}, \tag{41}$$

and

$$\text{Mod}\,\sigma_y^2(\tau) = (\frac{\Delta\nu_0}{\nu_0})^2\frac{\sin^6(\pi\ f_m n\tau_0)}{n^2(\pi\ f_m n\tau_0)^2\sin^2(\pi\ f_m\tau_0)}, \tag{42}$$

respectively. The effect of sinusoidal FM in both cases is 0 when τ equals the modulation period $T_m = f^{-1}{}_m$ or one of its multiples, since the modulating signal is completely averaged away. The largest value of Mod $\sigma_y(\tau)$ occurs when τ is $T_m/2$ or one of its odd multiples. Mod $\sigma_y(\tau)$ falls n times faster than $\sigma_y(\tau)$ for sinusoidal FM.

Linear Frequency Drift

When linear frequency drift exists (i.e., $y(t) = dt$), no tractable model exists for the power spectral density $S_y(f)$. Direct calculation in the time domain using Eqs. (17) and (34) yields

$$\sigma_y(\tau) = \frac{d}{\sqrt{2}}\tau \tag{43}$$

and

$$\text{Mod}\quad\sigma_y(\tau) = \frac{d}{\sqrt{2}}\tau, \tag{44}$$

respectively. Thus linear frequency drift yields τ^{+1} law for both $\sigma_y(\tau)$ and Mod $\sigma_y(\tau)$.

OTHER MEASURES OF FREQUENCY STABILITY

A number of other measures have been proposed and used during the past 35 years. Each measure has some advantages and limitations compared with the well-established power spectral density and the Allan variance.

Table 3. Conversion Chart for the Modified Allan Variance

Frequency Domain (Power Spectral Densities)	Time Domain (Allan Variances) Mod $\sigma_y^2(\tau)$
White x $S_y(f) = h_2 f^2$ $S_x(f) = h_2/(2\pi)^2$ $2\pi f_h \tau \gg 1$	$h_2 \cdot \dfrac{3f_h}{8\pi n \tau^2}$
Flicker x $S_y(f) = \mathrm{h}_1 f$ $S_x(f) = h_1/(2\pi)^2 f$ $2\pi f_h \tau \gg 1, 2\pi f_h T \gg 1$	$h_1 \cdot \dfrac{1}{4\pi^2 n^2 \tau^2}\left[3n \ln(2\pi f_h \tau) + \sum_{k=1}^{n-1}(n-k)\left\{4 \ln\left(\dfrac{n^2}{k^2} - 1\right) - \ln\left(\dfrac{4n^2}{k^2} - 1\right)\right\}\right]$
White y (random walk x) $S_y(f) = h_0$ $S_x(f) = h_0/(2\pi)_2 f^2$	$h_0 \cdot \dfrac{n^2 + 1}{4n^2\tau^2}$
Flicker y $S_y(f) = \dfrac{h_{-1}}{f}$ $S_x(f) = h_{-1}/(2\pi)^2 f^3$	$h_{-1} \cdot \dfrac{2 \ln 2}{n^2}\left[\dfrac{4n^2 - 3n + 1}{n^2} + \dfrac{1}{n^2 \ln 2}\sum_{k=1}^{n-1}(n-k)\left\{\dfrac{n}{2}[(k+2n)\ln(k+2n) - (k-2n)\ln(2n-k)] + \dfrac{1}{2}(k+n)(k=2n)\ln(k+n) + \dfrac{1}{2}(k-n)(k+2n)\ln\lvert k-n\rvert + 3k^2 \ln k - k\left[(n+2k)\ln\left(k+\dfrac{n}{2}\right) - (n-2k)\ln\left\lvert k - \dfrac{n}{2}\right\rvert\right]\right\}\right]$
Random walk y $S_y(f) = \dfrac{h_{-2}}{f^2}$ $S_x(f) = h_{-2}/(2\pi)^2 f^4$	$h_{-2} \cdot \dfrac{2\pi^2\tau}{3}\left(\dfrac{33}{40} + \dfrac{1}{8n^2} + \dfrac{1}{20n^4}\right)$

The Hadamard variance (11) has been developed for high-resolution spectral analysis of $y(t)$ from measurements of $\bar{y}_\mathrm{k}$. The high-pass variance (12) has been proposed through the transfer function approach and is defined by

$$\sigma^2(\tau) = \int_0^\infty S_y(f)|H(f)|^2 df. \tag{45}$$

It was shown that the Allan variance can be estimated by high-pass filtering the demodulated phase noise without using counting technique. A band-pass variance (12) has also been proposed to distinguish white and flicker phase noise processes. A filtered Allan variance (13) has been used to separate various noise processes.

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