STRESS–STRENGTH RELATIONS

It is easy to understand that a structure will fail if the load exceeds the capacity of the structure. If both the load and the capacity hold a single deterministic value, it is simple to figure out if the structure will fail. In the real world, the load on the structure rarely holds a deterministic value, and the

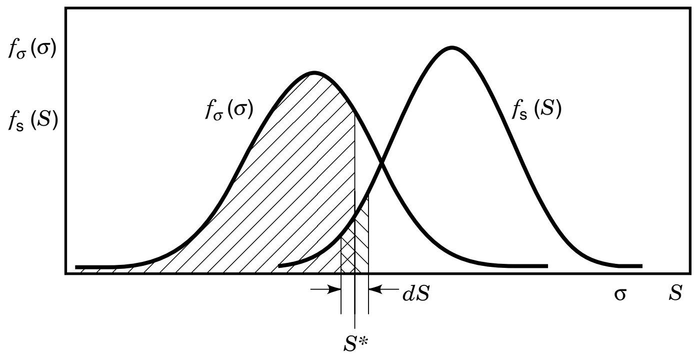

Figure 1. Stress–strength interference model.

same is true for the capacity of the same type of structure. The load and capacity are actually random variables or even stochastic processes. If we consider the load and the capacity to be random variables, we can use probability distributions to characterize the load and capacity. In order to calculate the reliability of the structure when the load and the capacity are random variables, a method is needed. One such method, the stress–strength interference model, can be used.

The stress–strength interference model is an important method used for structural reliability design and analysis. It can be used to calculate the probability that the load is smaller than the capacity. This probability is considered to be the reliability of the structure. The stress–strength interference model should really be called load-capacity interference model because of its broad scope of applications. For structures, the stress can be an applied load or load-induced response quantity that has the effect of causing the structure to fail (such as stress, force, moment, strain, deformation, pressure, or temperature). The strength can be the capacity of structures to withstand the applied load, such as yield strength, ultimate strength, yield moment, permissible deformation, allowable pressure, or temperature. The type of structure capacity depends on the type of the applied load and the failure criterion. In general, the principal driving force for structural failure under mechanical loading is the stress and the corresponding capacity is the strength. Hence, the stress–strength interference model was named. In this article, we will only refer to it as the stress–strength interference model.

MATHEMATICAL FORMULATION

Let us denote the stress and strength by σ and S, respectively. If the probability density functions of the stress and strength are $f_\sigma(\sigma)$ and $f_S(S)$, respectively, the reliability of the structure can then be calculated from $f_\sigma(\sigma)$ and $f_S(S)$. As shown in Fig. 1, the probability that the strength falls in the vicinity of S^* is $f_S(S^*)\,dS$, and the probability that the stress is smaller than S^* is $\int_{-\infty}^{S^*} f_\sigma(\sigma)d\sigma$. If these two events are independent, the probability that they occur at the same time is

$$dR = \int_{-\infty}^{S^*} f_\sigma(\sigma)\,d\sigma \cdot f_S(S)\,dS \tag{1}$$

For the entire distribution of the strength, the probability that the stress is smaller than the strength is

$$R = \int_{-\infty}^{\infty} f_S(S) \int_{-\infty}^{S} f_\sigma(\sigma)\,d\sigma\,dS \tag{2}$$

This probability is the reliability of the structure, and Eq. (2) is the mathematical expression of the stress–strength interference model.

If both the stress and strength are normally distributed, their probability density functions are

$$f_\sigma(\sigma) = \frac{1}{\sigma_\sigma\sqrt{2\pi}} e^{-\frac{(\sigma-\mu_\sigma)^2}{2\sigma_\sigma^2}} \tag{3}$$

and

$$f_S(S) = \frac{1}{\sigma_S\sqrt{2\pi}} e^{-\frac{(S-\mu_S)^2}{2\sigma_S^2}} \tag{4}$$

where μ_σ, σ_σ are the mean and standard deviation of the stress, respectively, and μ_S, σ_S are the mean and standard deviation of the strength, respectively.

Because the reliability of the structure is the probability that the stress is smaller than the strength, we have

$$R = P\{\sigma < S\} \tag{5}$$

Equation (5) can be rewritten into

$$R = P\{S - \sigma > 0\} = P\{\delta > 0\} \tag{6}$$

where $\delta = S - \sigma$. Here δ is also a normally distributed random variable. Its mean and standard deviation are

$$\mu_\delta = \mu_S - \mu_\sigma \tag{7}$$

and

$$\sigma_\delta = \sqrt{\sigma_S^2 + \sigma_\sigma^2} \tag{8}$$

Therefore, the reliability of the structure is

$$R = P\{\delta > 0\} = \int_0^\infty f_\delta(\delta)\,d\delta = \int_0^\infty \frac{1}{\sigma_\delta\sqrt{2\pi}} e^{-\frac{(\delta-\mu_\delta)^2}{2\sigma_\delta^2}}\,d\delta \tag{9}$$

where $f_\delta(\delta)$ is the probability density function of random variable δ. Let us introduce a new random variable Z through the transformation

$$z = \frac{\delta - \mu_\delta}{\sigma_\delta} \tag{10}$$

Equation (9) can be rewritten into

$$R = \int_{-\frac{\mu_\delta}{\sigma_\delta}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz = \int_{-\infty}^{\mu_\delta/\sigma_\delta} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\,dz = \phi\left(\frac{\mu_\delta}{\sigma_\delta}\right) \tag{11}$$

By substituting the expressions of μ_δ and σ_δ in Eqs. (7) and (8) into Eq. (11), we have

$$R = \phi\left(\frac{\mu_S - \mu_\sigma}{\sqrt{\sigma_S^2 + \sigma_\sigma^2}}\right) = \phi(\beta) \tag{12}$$

where the parameter β is known as the reliability index and $\phi(\beta)$ is the cumulative distribution function of the standard normal random variable. Values of the function $\phi(\beta)$ for some specific β are listed in Table 1. An example that illustrates the application of the preceding method follows.

Example 1. Consider a structural part that sustains a normally distributed static load. Under this load, the tensile stress of a critical location is also a normally distributed random variable. The mean and standard deviation for the tensile stress are 700 MPa and 200 MPa, respectively. The yield strength for the material of the part is also a normally distributed random variable with the mean of 1200 MPa and the standard deviation of 150 MPa.

To calculate the reliability, or the probability that the part will not yield, we calculate the reliability index

$$\beta = \frac{\mu_S - \mu_\sigma}{\sqrt{\sigma_S^2 + \sigma_\sigma^2}} = \frac{1200 - 700}{\sqrt{150^2 + 200^2}} = 2$$

From Table 1, the reliability of the part can be obtained as 0.9772.

In the preceding context, we use random variable $\delta = S - \sigma$ as the reliability performance function. When $\delta > 0$, the strength exceeds the stress, and the structure will survive. When $\delta < 0$, the strength is smaller than the stress, and the structure will fail. In general, the reliability of a structure depends on many relevant design as well as load parameters, which most likely are random variables. By denoting these random variables as $X_1, X_2, \ldots, X_n$, we can write the reliability performance function into $\delta = g(X_1, X_2, \ldots, X_n)$. When $\delta > 0$, the structure will survive. When $\delta < 0$, the structure will fail. $\delta = 0$ defines the boundary between the reliable and unreliable regions of the structure in the design parameter space. A reliability performance function can be an explicit or implicit function of basic random variables, and it can be in a simple or complicated form. If the joint probability density function for $X_1, X_2, \ldots, X_n$ is expressed as $f_X(x_1, x_2, \ldots, x_n)$, the reliability of structure can be calculated by

$$R = \iint\limits_{g(X_1, X_2, \ldots, X_n) > 0} \cdots \int f_X(x_1, X_2, \ldots, x_n)\, dx_1\, dx_2 \cdots dx_n \tag{13}$$

Because Eq. (13) can be used to calculate the reliability of a complicated structure, it is considered as a generalized stress–strength interference model.

If the random variables are statistically independent, the joint probability density function is the product of the individual probability density functions of random variables $X_1, X_2, \ldots, X_n$. Equation (2) is a special case of this. However, the random variables are generally not statistically independent. It is often difficult to obtain the joint probability density function of random variables. Even if this function is available, the calculation of the multiple integral in Eq. (13) is also formidable. If a closed form solution is not available, a numerical method has to be used to calculate the reliability. In some other cases, the random variables may not be described well by any standard probability distributions, and the probability density functions of the random variables can be represented in histograms or nonparametric statistical distributions only. Numerical methods also have to be used in these cases.

A straightforward numerical method to solve the stress–strength interference model in Eq. (2) or (13) is numerical integration. The procedure for numerical integration can be found in textbooks. In simple cases where only a few stress or strength related random variables are essential, it is easy to solve Eq. (13) by numerical integration. When the number of integration dimensions in Eq. (13) is relatively large, the computational time becomes too long. Numerical integration is no longer a practical method to solve Eq. (13). Monte Carlo simulation technique can be used in this case. The procedure for computing Eq. (13) by the Monte Carlo simulation can be described in the following.

First, a random number is generated for each variable $X_1, X_2, \ldots, X_n$ in Eq. (13) according to the joint probability density function $f_X(x_1, x_2, \ldots, x_n)$. Then, these random numbers are substituted into the reliability performance function $\delta = g(X_1, X_2, \ldots, X_n)$. It is known that the structure will survive when $\delta > 0$. Therefore, an estimate of the reliability can be obtained in the following equation by repeating the preceding sampling process

$$\overline{R} = \frac{N_{\delta>0}}{N} \tag{14}$$

where $N_{\delta>0}$ is the number of simulation cycles in which $\delta > 0$, and N is the total number of simulation cycles. As N approaches infinity, the estimated reliability $\overline{R}$ approaches the true reliability R. The accuracy that the estimated reliability $\overline{R}$ represents the true reliability R can be evaluated in terms of its variance. The variance of the estimated reliability can be computed by assuming each simulation cycle to constitute a Bernoulli trial. Therefore, the number of successes in N trials can be considered to follow a binomial distribution. The variance of the estimated reliability can be computed approximately as

$$\text{Var}(\overline{R}) = \frac{(1-\overline{R})\overline{R}}{N} \tag{15}$$

Another better alternative to measure the statistical accuracy of the estimated reliability is to use its coefficient of variation

Table 1. Reliability Index and Reliability

Reliability Index (β)	0	1	2	3	1.282	1.645	2.326	3.090	3.719
Reliability (R)	0.5	0.8413	0.9772	0.9987	0.9	0.95	0.99	0.999	0.9999

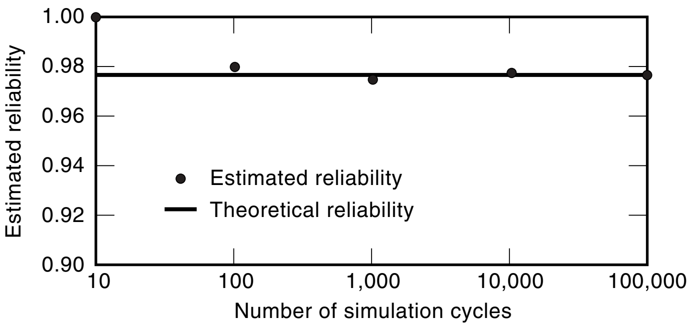

Figure 2. Estimated reliability according to Monte Carlo simulation.

$\mathrm{COV}(\overline{R})$, which can be calculated by

$$\mathrm{COV}(\overline{R}) = \frac{\sqrt{\dfrac{(1-\overline{R})\overline{R}}{N}}}{\overline{R}} \tag{16}$$

Example 2. Consider a structural part that sustains a normally distributed static load. Under this load, the tensile stress of a critical location is also a normally distributed random variable. The mean and standard deviation for the tensile stress are 700 MPa and 200 MPa, respectively. The yield strength for the material of the part is also a normally distributed random variable with the mean of 1200 MPa and the standard deviation of 150 MPa.

When direct Monte Carlo simulation is used to calculate the reliability of the part, the stress and strength random variables are randomly generated according to their respective probability distributions, and the reliability performance function $\delta = S - \sigma$ is evaluated. The estimate of the reliability can be obtained according to Eq. (14). For different numbers of simulation cycles, the estimated reliability is shown in Fig. 2. It is obvious that the estimated reliability converges to the theoretical reliability when the number of simulation cycles approaches infinity.

DISCUSSIONS

It is straightforward to obtain the reliability for a simple structure in a simple load case. In some cases, more advanced computational tools such as finite element analysis (FEA) have to be employed to provide the necessary computational framework for analyzing the reliability of complex structures due to the complexity in geometry, external loads, and nonlinear material behavior. The probabilistic finite element analysis (PFEA) is a rational way to evaluate the reliability of complex structures by combining the FEA with statistics and reliability methods. This is certainly one of the directions for future research.

As shown in Fig. 2, the accuracy of estimated reliability directly depends on the sample size in direct Monte Carlo simulation technique. To enhance the accuracy of the estimated reliability with moderate sample size, a more effective and efficient sampling method is necessary. Some good examples are stratified sampling method, importance-sampling method, Latin hypercube sampling method, and adaptive sampling method. In all these methods, the basic random variables are generated according to some carefully selected criteria so that simulation efficiency increases. This is another topic that needs to be addressed in the future.

The probability distributions of random variables are either obtained from experimental or field measurements or derived from other information. No matter how they are obtained, they are based on a finite amount of data. This introduces an uncertainty in the probability density functions. This uncertainty will be carried over to the calculation of reliability. The uncertainty associated with a calculated reliability can be expressed in terms of confidence level. There are researches that deal with the confidence of reliability. However, applicable results are available only in simple cases where both the stress and strength are normally distributed. How to build the confidence in general situations is certainly another challenge in the future.

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STRIPING. See INTERLEAVED STORAGE.
STRIPLINE CIRCUITS. See MICROWAVE CIRCUITS.