BURN-IN AND SCREENING

Burn-in and screening are methods that are extensively used in engineering. Their purpose is to eliminate weak items from a population. The distinction between the terms *burn-in* and *screening* is given later in this article. For the purposes of the article, we use burn-in as our catch-all phrase.

The population referred to previously usually consists of various engineering systems composed of items or parts that are arranged together to form the system. These items or parts, which we call components, operate for a certain amount of time until they fail, as do the systems composed of these components. The systems might be electronic systems such as circuit boards, and the components would be various types of chips and printed circuits. Alternately the systems considered could be mechanical, such as an air conditioner, where the components are condenser, fan, circuits, and the like. It is usually the case that within any population of components, there are strong components with long lifetimes and weak components with short lifetimes. To ensure that customers receive only the strong components, a manufacturer will subject all the components to tests where typical or severe use conditions are encountered. The weak components, in theory, will fail, leaving only the strong components. A similar type of test can be carried out after the components are assembled into systems. In addition to uncovering weak systems, this procedure is also used to uncover defects that were introduced during assembly. We use the term burn-in for all these tests. A basic issue is to find an optimal burn-in time.

In the following we give a general introduction to burn-in. We discuss terminology, burn-in distributions, performance criteria, cost functions associated with burn-in, mixture models, tail behavior for burn-in distribution, and a general optimization result.

TERMINOLOGY

Besides the term *burn-in,* closely related terms are *screen* and *environmental stress screening* (ESS). The AT&T Manual (1) defines a screen to be an application of some stress to 100% of the product to remove (or reduce the number of) defective or potentially defective units. Fuqua (2) concurs with the 100% but states that this may be an inspection and stress is not required. The same author describes ESS as a series of tests conducted under environmental stresses to disclose latent part and workmanship defects. Nelson (3) describes ESS as involving accelerated testing under a combination of random vibration and thermal cycling and shock. A more detailed description of ESS is given in Kuo, Chien, and Kim (4).

The AT&T Manual (1) describes burn-in as one effective method of screening (implying 100%) using two types of stress (temperature and electric field). Burn-in is described by Nelson (3) as running units under design or accelerated conditions for a suitable length of time. Tobias and Trindade (5) restrict burn-in to high stress only and require that it be done prior to shipment. The term *reliability audit* is used in the AT&T Manual (1) to describe the situation where a small number of complex systems are subjected to ordinary use and then mild stress conditions to, respectively, eliminate early system failures and accelerate aging so that weak systems fail. Furthermore, this type of audit is done to obtain data to compare equipment to certain standards set for it. In Bergman (6), burn-in is defined in a general way as a preusage condition of components performed to screen out the substandard components, often in a severe environment. The definition of burn-in in Jensen and Petersen (7) is basically the same as in Bergman.

In this article we use the term burn-in in a general way, similar to the usage of Bergman (6) and Jensen and Petersen (7). It will mean in the present setting, some preusage operation under which components or systems undergo normal or stressed conditions. It can involve either 100% of the items or some smaller subgroup (especially for complex systems or subsystems), and it is not limited to eliminating weak components.

A good introduction to some of the statistical ideas in burn-in are contained in Jensen and Petersen (7). Some of the engineering as well as statistical ideas are contained in Fuqua (2) and Tobias and Trindade (5). A useful reference is the AT&T Manual (1). Two papers that review the engineering literature are Kuo and Kuo (8) and Leemis and Beneke (9), while a paper by Block and Savits (10) reviews the statistics literature. A book by Kuo et al. (4) contains very up-to-date statistical and engineering material.

BURN-IN DISTRIBUTIONS

The type of component that will benefit from burn-in can be described by the shape of its failure rate function. If the failure rate function is increasing, the component whose lifetime has this failure rate is wearing out. Thus, if such a component is subject to burn-in, some of its more reliable life is being used, and burn-in will not in general be beneficial. For burn-in to be effective, components should have high failure rates initially and then improve. Burn-in for such distributions essentially eliminates the part of the lifetime where there is a high chance of failure. The class of lifetimes with bathtub-shaped failure rates has this property. These distributions have high failure rates initially (the infancy period), decrease to a constant (the middle life), and then eventually increase (old age), which represents wearout. As indicated by the language in the parentheses, these distributions are thought to describe human life and other biological systems. However, certain electronic and mechanical lifetimes can also be modeled by these distributions. This distribution is appropriate for burn-in because burn-in eliminates the high-failure infancy period, leaving a lifetime that begins near its middle life period. We generally refer to the end of the infancy period as the first change point and denote it by t_1. Similarly the beginning of the old age period is called the second change point and is denoted by t_2. See Fig. 1 for a typical bathtub curve and the effect of burn-in.

There are physical reasons for using bathtub-shaped distributions to describe many systems and components. As pointed out by Jensen and Petersen (7), many industrial populations are heterogeneous, and there are only a small number of subpopulations. Although members of these subpopulations may not have bathtub-shaped failure rates, sampling

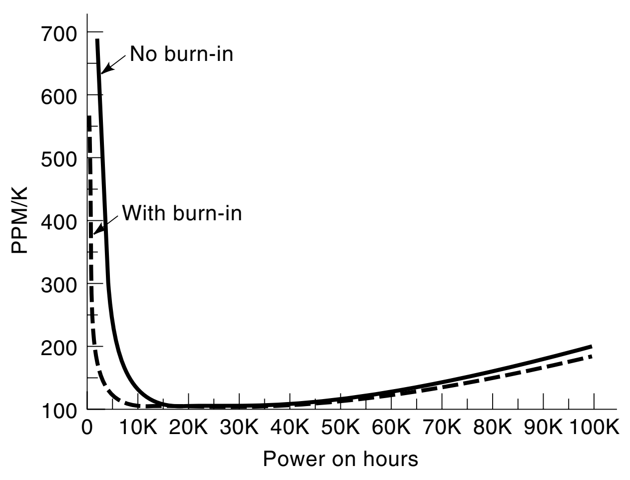

Figure 1. A typical bathtub curve and effect of burn-in. Reproduced from Tobias and Trindade (5) with permission.

from them produces a mixture of subpopulations, and these mixtures often have bathtub-shaped failure rates.

Various papers have appeared in the statistical literature providing models and formulas for bathtub-shaped failure rates. See Rajarshi and Rajarshi (11) for a review of this topic and many references. One method of obtaining some of these distributions is by mixing standard life distributions such as the exponential, Weibull, and gamma. The AT&T Manual (1) gives another model, called the AT&T model, for the failure rate of an electronics component. The initial part of the lifetime is modeled by a Weibull with decreasing failure rate, and the later part is modeled by an exponential. This model does not provide for wearout, but the Manual explains that the AT&T electronic equipment tends not to wear out before it is replaced. This model has been used extensively by Kuo and his coauthors [see Kuo et al. (4) for a discussion]. This model is also called the Weibull-exponential model in the statistical literature.

PERFORMANCE CRITERIA

The purpose of burn-in is to improve the quality of products after they have been produced. The quality of products can be measured via various performance characteristics. Throughout most of this article, we assume that the failure rate function of a product exhibits a bathtub shape and let t_1 denote its first change point (i.e., the first point the failure rate stops decreasing). Notice that both increasing and decreasing failure rate functions are special cases of a failure rate function with bathtub-shaped failure rate. We will consider the following optimization problems pertinent to product performance. References for many of the following criteria are contained in Block and Savits (10). One recent reference is Mi (12) in which availability criteria are discussed.

Maximizing Conditional Survival Probability

Let $F(t)$ be the distribution function of a product and $\overline{F}(t) = 1 - F(t)$ its survival function. A product surviving the burn-in procedure which lasts for time b has a conditional survival function $\overline{F}_b(t) \equiv \overline{F}(b + t)/\overline{F}(b)$ for $t > 0$. In practice, we may want this product to operate for a fixed mission time τ without failure. The probability of this event is given by $\overline{F}_b(\tau)$, which is in fact the conditional probability for the product to last for at least τ units of time given that it has survived b units of time. Certainly, we are interested in the optimal burn-in time, say b^*, that will maximize the preceding conditional survival probability. That is, we need to find b^*, which maximizes $\overline{F}(b + \tau)/\overline{F}(b)$.

Maximizing Mean Residual Life

The mean residual life at time t of a product is the mean of the residual life of a product that has survived time t. To be more precise, letting X be the random life of the product, then the value of the mean residual life at time t is $E(X - t|X \geq t)$. Denoting this function by $\mu(t)$, it can be shown that

$$\mu(t) = \frac{\int_t^\infty \overline{F}(x)\,dx}{\overline{F}(t)}$$

From the definition of mean residual life and the preceding expression, we see that at time $t = 0$, $\mu(0)$ is exactly the same as the mean life of the product. The mean residual life is certainly a natural measure of product performance. In practice, we often want the product to have a long mean life in field operation. This goal can be achieved by the burn-in procedure, and it is equivalent to finding the optimal burn-in time b^* that will maximize $\mu(b)$.

Minimizing the Mean Number of Failures Before Completion of a Mission

Suppose that in field operation we use components that have survived the same burn-in procedure with time b. All these components have the same distribution function $F_b(t)$. At failure, the failed components will be replaced by independent ones, and this process continues until it reaches a predetermined mission time τ. The number of failures in the interval $[0, \tau]$ is also a measure of product performance. If we want to have fewer failures in this interval, then we can again appeal to burn-in. Denote the number of failures in the interval by $N_b(\tau)$, the optimal burn-in time b^* is obtained by maximizing $E[N_b(\tau)]$.

Maximizing Warranty Period

Let $F(t)$ be a lifetime distribution function and α a given percentage. We define the α-percentile residual life function by

$$q_\alpha \equiv \overline{F}^{-1}(\alpha) \equiv \inf\{t \geq 0 : \overline{F}(t) \leq 1 - \alpha\}$$

This quantity can be viewed as the warranty period for which at most α-percent of components following the distribution will fail. We use $F_b(t)$ to denote the distribution function of components that have survived burn-in time b. According to the definition of α-percentile residual life function, we can similarly define

$$q_\alpha(b) \equiv F_b^{-1}(\alpha) \equiv \inf\{t \geq 0 : \overline{F}_b(t) \leq 1 - \alpha\}$$

One problem is to determine how long we should burn in components so that the associated warranty period will be maximized. The corresponding optimal burn-in time is thus determined by maximizing $q_\alpha(b)$.

Maximizing Stable Point Availability

Suppose that in field operation independent and identical components are used sequentially and one at a time. At each failure, the failed component is replaced by another new one, with negligible detection and replacement times. We further assume that the random times needed for completing these replacements are independent of each other and have the same distribution. For any given time t, the probability that the system is in working state is called the instantaneous availability of the system. The limit of the instantaneous availability as $t \to \infty$ is called the stable point availability. Stable point availability is a measure of system performance and can be expressed as the ratio of the mean lifetime of the system over the sum of the mean lifetime of the system and the mean replacement time. Therefore, in order to increase the stable point availability, we need only to increase the mean lifetime of the system if the mean replacement time is fixed. Suppose that we burn in the system for time b. The mean life of the system used in field is thus the mean residual life $\mu(b)$ mentioned in the preceding subsection on maximizing mean residual life. Hence, this optimization problem is equivalent to maximizing $\mu(b)$.

Maximizing Stable Interval Availability

The assumptions made in the previous subsection are retained here. An interesting and useful quantity is interval availability. For any fixed $w \geq 0$ and time t, the corresponding interval availability is defined as the probability that the system is working failure-free in the entire interval $[t, t + w]$. Except for a very few special cases, there is not closed form expression for interval availability. However, from an engineering point of view, the long run behavior of interval availability is good enough for measuring the performance of the system. The limit of interval availability as $t \to \infty$ is called the stable interval availability of the system and is denoted by $A(w)$. If the cumulative distributions of the system life and replacement time are denoted by $H(t)$ and $G(t)$, respectively, then the expression for $A(w)$ is given by

$$A(w) = \frac{1}{\int_0^\infty \overline{H}(t)\,dt + \int_0^\infty \overline{G}(t)\,dt} \int_w^\infty \overline{H}(t)\,dt$$

Now, suppose that we use systems that have survived the same burn-in procedure for b units of time. In this case, the distribution of the system life is $F_b(t)$. If we still assume the distribution of replacement times is $G(t)$, then the stable interval availability has the expression

$$A_b(w) = \frac{1}{\mu_F(b) + \nu} \int_w^\infty \overline{F}_b(t)\,dt$$

where

$$\mu_F(b) = \frac{\int_b^\infty \overline{F}(t)\,dt}{\overline{F}(b)} \quad \text{and} \quad \nu = \int_0^\infty \overline{G}(t)\,dt$$

A natural question is whether burn-in can be applied to increase stable interval availability, and if so, how long should the burn-in procedure last. That is, we need to find the optimal time b^* maximizes $A_b(w)$. For all these performance criteria, it has been proved that if the underlying distributions $F(t)$ exhibits a bathtub-shaped failure rate, then the optimal burn-in time b^* satisfies $b^* \leq t_1$ where t_1 is the first change point of the failure rate function of $F(t)$.

COST FUNCTIONS

It is appropriate to use the previously mentioned performance criteria without considering cost resulting from burn-in when those performance criteria are extremely important and cost is not a major issue. There are many other circumstances, however, in which cost must be taken into consideration for determining the optimal burn-in time. This is the topic of the present section. In most of the following cases, it is shown that if a bathtub-shaped failure rate is assumed, the optimal burn-in time (i.e., the time that minimizes cost) must occur at or before the first change-point t_1.

Clarotti and Spizzichino Cost Function

The following is a cost function due to Clarotti and Spizzichino (13). A component is burned-in for b units of time. If it fails during the procedure, a cost c_0 is incurred. If it survives burn-in, then it will be put into use in the field; in this case, if it fails before a mission time τ, then an additional cost $C > c_0$ is incurred; and if it successfully operates during the mission time, then a gain of K is received. Consequently, if the underlying distribution is $F(t)$, then the mean cost function is

$$c_1(b) = c_0F(b) + C[F(b+\tau) - F(b)] - K\overline{F}(b+\tau)$$

In the following discussion, we assume the total cost is the sum of the cost of burn-in and the cost incurred in field application of the burned-in components.

Cost Resulting from Burn-In

An attempt is made to burn-in a new component for b units of time. If the component fails before time b, then it is replaced by a new one (or, if repairable, by a good as new one) with a cost c_s, and the burn-in procedure with the same time b begins anew and continues until a component that has survived the burn-in period b is obtained. It is also assumed that the cost of burn-in per unit time is c_0. Denoting the distribution function of a new component by $F(t)$, it can be shown that the mean cost resulting from this burn-in procedure is given by

$$k(b) \equiv c_0 \frac{\int_0^b \overline{F}(t)\,dt}{\overline{F}(b)} + \frac{c_sF(b)}{\overline{F}(b)}$$

It is easy to show that $k(b)$ is an increasing function of b.

Burn-In Cost Plus Cost and Gain for Mission Time Period

Suppose that a component that has been burned-in and put into field operation fails before mission time τ. A cost C is then incurred. If the component does not fail, a gain of K is obtained. The total mean cost is expressed as

$$c_2(b) = k(b) + C\frac{F(b+\tau) - F(b)}{\overline{F}(b)} - K\frac{\overline{F}(b+\tau)}{\overline{F}(b)}$$

Burn-In Cost and Gain Proportional to the Mean Life in Field Use

Suppose that the operation of a burned-in component incurs a gain proportional to its mean life. The mean total cost is then given by

$$c_3(b) = k(b) - K\frac{\int_b^\infty \overline{F}(t)\,dt}{\overline{F}(b)}$$

where K is the proportionality constant. See Mi (14) for an analysis of $c_2(b)$ and $c_3(b)$.

Burn-In and Warranty Policy

Consider either a failure-free warranty policy or rebate warranty policy. Suppose burned-in products are sold along with a warranty that has a fixed warranty period T. A manufacturer is interested in finding the optimal burn-in time b^* which can minimize the total mean cost incurred by both burn-in and warranty. This question has been discussed by Mi (15).

Burn-In and Age Replacement Policy

In this case we consider both burn-in and maintenance policies. In particular, we will use an age replacement policy T. Let c_f denote the cost incurred for each failure in field operation, and let $c_a < c_f$ the cost incurred for each nonfailed component that is replaced at age T in field operation. Combining these costs with the burn-in cost $k(b)$, we obtain the long-run average cost per unit time as a function of the burn-in time b and the age replacement policy T

$$c(b,T) = \frac{c_fF_b(T) + c_a\overline{F}_b(T) + k(b)}{\int_0^T \overline{F}_b(t)\,dt}$$

The case where b and T are optimized simultaneously has been considered by Mi (16).

Other Cost Functions

In the review papers of Kuo and Kuo (8), Leemis and Beneke (9), and the book by Kuo et al. (4), many other cost functions are mentioned.

GENERAL OPTIMIZATION RESULT

In the previous sections we listed several criteria that could be used for deciding how long burn-in should continue. These are but a small sample of the various utility or objective functions that have been considered in the burn-in literature. As has been noted, a striking feature among these is that the optimal burn-in time b^* generally occurs at or before the first change point t_1 of the underlying bathtub distribution $F(t)$.

This result is intuitively satisfying and is believed to hold true for any "reasonable" objective function. It does also have important implications because it provides an upper bound for burn-in. A recent result of Block et al. (17) attempts to capture the essentials of this "folklore."

Consider an objective function $C(b)$, which can be decomposed into two parts: $C(b) = C_1(b) - C_2(b)$. We think of $C_1(b)$ as the "cost" of burn-in, while $C_2(b)$ represents the gain due to burn-in. An immediate observation is that if $C_1(b)$ increases in $b \geq t_1$ and $C_2(b)$ decreases in $b \geq t_1$, then $C(b)$ is increasing in b $\geq t_1$, and hence its minimum value must occur no later than t_1.

This framework simplifies and unifies many of the previously obtained optimization results. Maximization problems and objective functions of the form $C(b) = [C_1(b)]/[C_2(b)]$ can be easily transformed into the preceding prototype by taking negatives and/or logarithms. Jong (18) has shown that most of the objective functions studied in the literature can be cast in this framework.

PRESERVATION RESULTS

In the framework of the previous section, we were interested in optimizing utility functions of the form $C(b) = C_1(b) -$

$C_2(b)$. In most situations, the functions $C_1(b)$, $C_2(b)$ are themselves expressible as monotone functionals of simpler well-known objects. For example, we can regard the mean residual life function $\mu(b)$ as a functional Φ defined on the class of life distributions $\mu(b) = \Phi(F_b)$, where

$$\Phi(G) = \int_0^\infty \overline{G}(t)\,dt$$

and F_b is the distribution function of the burned-in unit. Furthermore, Φ is monotone in the sense that if G_1 is stochastically less that G_2 [i.e., if $\overline{G}_1(t) \le \overline{G}_2(t)$ for all $t \ge 0$, then $\Phi(G_1) \le \Phi(G_2)$].

Thus it is important to have a collection of building blocks for which properties are known. In particular, it is important to understand the ramifications of the class of bathtub distributions.

In Block et al. (17), two different characterizations of a bathtub-shaped function are given in terms of their sign-change properties. One application gives that if F has a bathtub-shaped failure rate function with the first change-point t_1, then F_b is stochastically decreasing in $b \ge t_1$, that is, F_b is stochastically larger than F_c for $t_1 \le b \le c$. Consequently, the mean residual life function $\mu(b) = \Phi(F_b)$ is decreasing in $b \ge t_1$ and thus its maximum value occurs at or prior to t_1.

In a recent paper of Block et al. (19), these authors investigate preservation properties of bathtub-shaped functions. A main result in that paper gives conditions under which a function of the form $G(t) = N(t)/D(t)$ inherits the bathtub-shaped property from the related function $\eta(t) = N'(t)/D'(t)$. Furthermore, the change point of G is bounded by the change point of η. Many of the quantities of interest in reliability have this form. Generally the associated function $\eta(t)$ is easier to work with. For example, in the case of the mean residual life function, $\eta(t) = 1/r(t)$, where $r(t)$ is the failure rate function of the distribution $F(t)$. It is shown that if $r(t)$ has a bathtub shape, then $\mu(t)$ has an upside-down bathtub shape; moreover, the first change point of $\mu(t)$ occurs no later than that of $r(t)$. Applications to other reliability functions include the variance residual life function, the uncertainty function, and the coefficient of variation function. A detailed discussion of a burn-in criterion using the coefficient of variation as the objective function is also given.

BURN-IN AND MIXTURE MODELS

Mixtures are important not only because they give an explanation of a bathtub-shaped failure rate but also because they reflect practical concerns as described in the introduction. For the model of a general mixture, let S be the index set, $F(t, \lambda)$, $\lambda \in S$ be the distribution of any λ subpopulation, and let P be a probability measure on S. The distribution of the mixture is then given by

$$F(t) = \int_S F(t,\lambda)P(d\lambda)$$

Clarotti and Spizzichino (13) considered the special case of mixtures of exponentials. That is, in their model, $F(t, \lambda) = 1 - \exp(-\lambda t)$ for every $\lambda \in S$. Combining their cost function $c_1(b)$ with this mixture model, they derived some interesting results. One result gives that the limiting behavior of the failure rate of the mixture is $\alpha = \inf_{\lambda\in S}\lambda$. Another result shows that the likelihood ratio ordering between two mixing probability measures P_1 and P_2 implies an ordering of the associated optimal burn-in times b_1^* and b_2^*. Block et al. (20) extend these results to the general mixture model. Generally a population of products consists of different groups of components that are not distinguishable but that have different qualities. Accordingly, we can say that the entire population consists of subpopulations of different strengths. In this general setting, these authors show the intuitively appealing result that the strongest subpopulation will eventually dominate the entire population.

In the following subsections we explore three different criteria for determining the strongest subpopulation. More details on this are contained in Mi (21).

Failure Rate Criterion

Suppose that each $F(t, \lambda)$ has failure rate function $r(t, \lambda)$ and the limit $a(\lambda) = \lim_{t\to\infty} \gamma(t, \lambda)$ exists. Under certain conditions, Block et al. (20) show that the failure rate function $r(t)$ of the mixture converges as t goes to infinity; more precisely,

$$\lim_{t\to\infty} r(t) = \inf_{\lambda\in S} a(\lambda) \equiv \alpha$$

In this case we can use failure rate as a criterion: for any fixed failure rate level $c > \alpha$, designate a subpopulation $\{\lambda \in S : a(\lambda) < c\}$ as the strong subpopulation and $\{\lambda \in S : a(\lambda) \ge c\}$ as the weak subpopulation. It can be shown that at time t the proportion of the strong subpopulation that survived time t is given by

$$M_t(\{\lambda \in S : a(\lambda) < c\}) = \frac{\int_{\{\lambda\in S\,:\,a(\lambda)<c\}} \overline{F}(t,\lambda)P(d\lambda)}{\int_S \overline{F}(t,\lambda)P(d\lambda)}$$

The results obtained in Block et al. (20) imply that for any $c > \alpha$ the preceding proportion $M_t(\{\lambda \in S : a(\lambda) < c\})$ has limit one as $t \to \infty$.

Mean Residual Life Criterion

Let $\mu(t, \lambda)$ denote the mean residual life function for each $\lambda \in S$ and assume that the limit $b(\lambda) = \lim_{t\to\infty}\mu(t, \lambda)$ exists. For any fixed level $c < \beta \equiv \sup_{\lambda\in S} b(\lambda)$, we designate the subpopulation $\{\lambda \in S : b(\lambda) > c\}$ as the strong subpopulation and $\{\lambda \in S : b(\lambda) \le c\}$ as the weak subpopulation. As before, Mi (21) shows that the proportion of the strong subpopulation which has survived time t has limit one as $t \to \infty$; in other words,

$$\lim_{t\to\infty} M_t(\{\lambda \in S : b(\lambda) > c\}) = 1$$

for any $c < \beta \equiv \sup_{\lambda\in S} b(\lambda)$.

Conditional Survival Probability Criterion

Mi (21) also considers using the conditional survival function $\overline{F}(t + x, \lambda)/\overline{F}(t, \lambda)$ to define the notion of a strong subpopulation and uses a result of Rojo (22) to obtain results similar to those of the preceding two subsections.

EVENTUAL MONOTONICITY OF FAILURE RATES OF MIXTURES

As mentioned in an earlier section, burn-in is usually applied to distributions that arise as mixtures. These distributions often have failure rates that are bathtub-shaped. Consequently, the study of the behavior of mixture distributions and in particular the question of when their failure rates are bathtub-shaped are of great importance. Preliminary investigations have focused on the tails of the failure rates of mixture distributions. For failure rate functions to be bathtub-shaped, they must at least initially decrease and eventually increase. Burn-in is intimately connected with an initial decrease in the failure rate function. Because most components or systems are subject to eventual wearout, intuitively the right tail of the failure rate should eventually be increasing. In this section, we focus on the right tail of the distribution.

One of the earlier studies of mixtures of distributions was Proschan (23). In this study, the question of why the lifetimes of cooling systems of aircraft had failure rates that decreased was discussed. It turns out that these lifetimes came from several exponential populations that, of course, had constant failure rates. Collectively, the overall failure rate appeared to be decreasing. The explanation given by the author was that the resulting population consisted of a mixture of exponentials and consequently had a decreasing failure rate. An intuitive explanation is that there is a stronger component and weaker components and over the course of time the effect of the weaker components dissipates and the stronger component takes over. In terms of failure rates, the overall failure rate decreases to the stronger failure rate. Although this result has since been observed in special cases, one of the first general results of this type occurs in Block et al. (20) for continuous distributions. The result of these authors, discussed in a previous section, is that the failure rate of a mixture eventually approaches the limiting failure rate of the strongest component. A companion result appears in Mi (24) for discrete distributions.

Gurland and Sethuraman (25, 26) made the observation that mixtures that contained distributions with rapidly increasing failure rates could be eventually decreasing. In the following we will describe some results of Block and Joe (27), which put all the preceding results in context and give general conditions for mixtures to have eventually increasing or decreasing failure rates. The reasons for the importance of this work follow: (1) it is important to know the behavior that occurs when populations are pooled (this pooling can occur naturally as described previously or can be done by statisticians to increase sample size), and (2) it is useful, for modeling purposes, to have available a distribution with a particular failure shape (e.g., a bathtub-shaped distribution).

Eventually Decreasing Mixtures

More general versions of the observations of Gurland and Sethuraman (25, 26) were obtained by Block and Joe (27). Many of the results of Gurland and Sethuraman have to do with mixtures of exponential distributions and other lifetime distributions having an increasing failure rate. These mixtures turn out to have eventually decreasing failure rates. The result that we mention later is from the paper of Block and Joe (27) and represents a number of results contained in Theorems 2.1 and 2.2 of that paper.

These authors consider a mixture of two lifetime distributions with densities f_1, f_2, survival functions $\overline{F}_1$, $\overline{F}_2$, failure rate functions r_1, r_2, and weights p and $q = 1 - p$ where $0 < p < 1$. The mixed density is then written as

$$f(t) = pf_1(t) + (1-p)f_2(t)$$

and the mixed survival function is

$$\overline{F}(t) = p\overline{F}_1(t) + (1-p)\overline{F}_2(t)$$

The failure rate of the mixture is $r(t) = f(t)/\overline{F}(t)$, which is a complicated function of the individual failure rates $r_1(t)$ and $r_2(t)$. It is assumed that the second failure rate is eventually stronger than the first in the sense that $r_1(t) \geq r_2(t)$ for all large values of t. They also make some technical assumptions to ensure that the failure rates behave like ratios of polynomial functions because almost every major life distribution has this property. Under appropriate conditions, if

$$\frac{r_1(t)}{r_2(t)} \text{ is increasing in } t$$

and has an infinite limit, then

$$\frac{r(t)}{r_2(t)} \text{ is decreasing in } t$$

Notice that, in particular, if $r_2(t)$ is also decreasing, then $r(t)$ is decreasing, which is the desired result.

For an example of the use of the preceding, consider two Weibull distributions with failure rates $r_i(t) = \theta_i\gamma_i t^{\gamma_i}$, $i = 1, 2$ with the second failure rate being stronger eventually (i.e., $\gamma_1 \geq \gamma_2$). Then

$$\frac{r_1(t)}{r_2(t)} = \frac{\theta_1\gamma_1}{\theta_2\gamma_2} t^{\gamma_1-\gamma_2}$$

is increasing in t and has an infinite limit. By the preceding equation $r(t)/r_2(t)$ is decreasing in t. For $\gamma_2 < 1$, $r_2(t)$ has a decreasing failure rate and so the mixture $r(t)$ is eventually decreasing. This result is not so easy to verify directly.

Conditions Under Which Mixtures Experience Wearout

Another result obtained by Block and Joe (27) gives that for a wide variety of failure distributions, a mixture eventually inherits the monotonicity of its strongest component. For example, if the strongest component is eventually increasing, so is the mixture. This is equivalent to saying the mixture experiences wearout. Notice that by way of contrast the result of the previous example gives that the mixture is eventually decreasing, which is not what one would expect of many physical systems.

Most standard failure rate distributions such as the Weibull, the gamma, and the lognormal have failure rates that approach constant or infinite limits at a reasonable rate as time increases. We categorize these distributions by saying that their failure rates approach a limit at polynomial rate. See Block and Joe (27) for a more precise definition of polynomial rate and for the definition of the following distribution.

A truncated extreme distribution has a failure rate that approaches a limit much more quickly than the preceding failure rates. This failure rate is said to have an exponential rate. Using this terminology we can state the main result of Block and Joe (27).

Consider a mixture model with two components in which the second component is stronger and both components have monotone failure rates that approach constants r_1 and r_2 at polynomial rates with $r_1 > r_2$. Under a technical condition on the derivatives of the failure rates, the failure rate of the mixture has eventual monotonicity in the same direction as the strongest component.

Consider the mixture of two gamma distributions with densities proportional to $t^{\beta_i - 1} \exp(-\alpha_i t)$ for $i = 1, 2$. Assume that $\alpha_i > 1$ for $i = 1, 2$ so that the distributions have increasing failure rates and also that $\alpha_1 > \alpha_2$ so that the second distribution is stronger than the first. The result mentioned previously gives that any mixture of these two distributions has eventually an increasing failure rate.

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