SERVOMECHANISMS

In the field of automatic control and control system technology, it is often the case that one may wish to modify the behavior of a system that is influenced by external disturbances so that certain desirable properties occur. For example, when heating a room subject to outside temperature variations, one may need to increase or decrease the input fuel rate to the heater of the room in order to maintain a constant indoor temperature. A device that achieves this task is called a servomechanism controller.

This article reviews the theory of servomechanism control and begins with an overview description of the mathematical modeling of physical systems using state-space methods (1,2). A number of problems that can be considered when using such model representations are then presented. In particular, the problem of dynamic inversion (3,4) of a system is considered, in which it is desired to find a controller such that the resulting controller cascade system has a unity diagonal behavior. The general problem of robust control and uncertain linearized systems is then considered, dealing with the premise that any mathematical model of a physical system only approximates the actual behavior of the system. In this case, given a stable linear time-invariant (LTI) model with real parameter perturbations, one problem of interest is to determine the largest parametric perturbation that may occur such that the resultant perturbed system remains stable; this is called the real stability radius problem (5).

The robust servomechanism problem is then considered (6–11) in which it is desired to find a controller for the system such that

1. the resultant closed loop system is stable, and
2. asymptotic error regulation and tracking occur for a specified class of disturbance and reference signals.

This article concludes with the study of an adaptive servomechanism control problem using switching mechanisms (12–21), in which very little a priori knowledge of the mathematical model of the system to be controlled is assumed to be known.

NOTATION

The following mathematical notation will be used in a fairly consistent manner throughout this article.

Let $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$, and $\mathbb{C}$ denote, respectively, the set of real, positive real, natural, and complex numbers. $\mathbb{R}^n$ will be the n-dimensional real vector space, $\mathbb{R}^{m\times n}$ will be the set of $m \times n$ real matrices, $\mathbb{C}^-$ ($\mathbb{C}^+$) will be the set of complex numbers with strictly negative (positive) real parts, and $\mathbb{C}^0$ will be the set of complex numbers lying strictly on the imaginary axis. For any $x, y \in \mathbb{N}$,

$$x \bmod y := x - \text{floor}\left(\frac{x}{y}\right) y$$

where floor (∗) rounds the expression (∗) down to the nearest integer.

With $x \in \mathbb{R}^n$, denote its ∞-norm to be

$$\|x\| := \max_{1\le i\le n} |x_i|$$

For any arbitrary $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times n}$, let eig($A$) denote the eigenvalues of A, and let rank(B) denote the rank of B. Matrix A is said to be stable if eig(A) $\subset \mathbb{C}^-$ and unstable otherwise. In addition, a piecewise continuous function $f : [0, \infty) \to \mathbb{R}^n$ will be said to lie in $\mathscr{L}_\infty$ ($f \in \mathscr{L}_\infty$) if

$$\|f\| := \sup_{t\ge 0} \|f(t)\|$$

is finite.

STATE-SPACE MODELS

In the general representation of linear time-invariant, finite dimensional systems, the state variable model, or state-space model, is a set of first-order coupled differential equations written in vector matrix form. This representation preserves the input-output relationship of the Laplace transfer function, while representing internal characteristics of the modeled system.

As an example, consider the general dynamic system **S** given in Fig. 1, which has m inputs and r outputs. The state variables of dynamic system **S** are defined as the set of numbers

$$x_1(t), x_2(t), \ldots, x_n(t)$$

that contain sufficient information about the history of the system such that, if the values of

$$x_1, x_2, \ldots, x_n$$

are known at any time t_0 together with knowledge of the system input for $t_0 \le t \le t_1$, then the evolution of the n states and the r outputs of the system for $t_0 \le t \le t_1$ are completely defined. Hence, for the continuous-time case, dynamic system **S** can be represented by a set of n first-order coupled differential equations given by

$$\frac{dx_i(t)}{dt} = f_i[x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_m(t), t], \quad i \in \{1, 2, \ldots, n\} \tag{1a}$$

where $x_1(t)$, $x_2(t)$, . . ., $x_n(t)$ are the n state variables of the system, $u_1(t)$, $u_2(t)$, . . ., $u_m(t)$ are the m input variables to the system, and f_1, f_2, . . ., f_n are n scalar-valued functions. In addition, because the state variables completely define the dynamic behavior of the system, the r outputs of system **S** can also be expressed as

$$y_k(t) = g_k[x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t), \ldots, u_m(t), t], \quad k \in \{1, 2, \ldots, r\} \tag{1b}$$

where g_1, g_2, . . ., g_r are r scalar-valued functions.

Matrix Representation

Because it is often mathematically more convenient to deal with vectors, define the state, output, and input vectors to be,

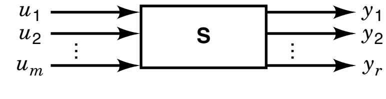

Figure 1. A block diagram of dynamic system **S** with m inputs and r outputs.

respectively,

$$x(t) := [x_1(t) \quad x_2(t) \quad \cdots \quad x_n(t)]^T$$
$$y(t) := [y_1(t) \quad y_2(t) \quad \cdots \quad y_r(t)]^T$$
$$u(t) := [u_1(t) \quad u_2(t) \quad \cdots \quad u_m(t)]^T$$

where $(x, y, u) \in \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}^m$. Then Eq. (1) also can be expressed equivalently in the vector matrix form given by

$$\frac{dx(t)}{dt} = f(x, u, t)$$
$$y(t) = g(x, u, t)$$

where $f \in \mathbb{R}^n$ and $g \in \mathbb{R}^r$ are vector-valued functions (1,2).

In practice, however, the assumption that a system is linear and time-invariant often is made; in this case, the state equations for a multi-input multi-output (MIMO) LTI continuous-time system can be given by

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2a}$$
$$y(t) = Cx(t) + Du(t) \tag{2b}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^r$ is the output vector, and $(C, A, B, D) \in \mathbb{R}^{r\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{r\times m}$. Moreover, it should be noted that the state-space representation for any LTI MIMO system is not unique because, upon defining $z(t) := Tx(t)$, where $T^{-1} \in \mathbb{R}^{n\times n}$ is assumed to exist, Eq. (2) can be written alternatively as

$$\dot{z}(t) = TAT^{-1}z(t) + TBu(t) \tag{3a}$$
$$y(t) = CT^{-1}z(t) + Du(t) \tag{3b}$$

For most physical systems, D is equal to zero because a nonzero value of D indicates that at least one direct path between one of the m system inputs and one of the r system outputs exists. In addition, for the case of single-input single-output (SISO) systems, $m = r = 1$, and matrices B and C reduce to column and row vectors, respectively.

Laplace Transform Solution

Assuming that the Laplace transform of $u(t)$ exists, the Laplace transform of Eq. (2) can be taken to yield

$$sX(s) - x_0 = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

where $X(s)$, $Y(s)$, and $U(s)$ denote the vector Laplace transforms of $x(t)$, $y(t)$, and $u(t)$, respectively, and where the initial state $x(t_0)$ is denoted as x_0. However, because

$$X(s) = (sI - A)^{-1}BU(s) + (sI - A)^{-1}x_0 \tag{4a}$$

therefore

$$Y(s) = [D + C(sI - A)^{-1}B]U(s) + C(sI - A)^{-1}x_0 \tag{4b}$$

In Eq. (4b), $[D + C(sI - A)^{-1}B]$ is referred to as the transfer function matrix of the LTI system **S**, whereas $C(sI - A)^{-1}$ represents the initial condition matrix.

Remark 1. Consider Eq. (3), and define $z_0 := Tx(t_0)$, where $T^{-1} \in \mathbb{R}^{n\times n}$ is assumed to exist. The Laplace transform of Eq. (3) then can be written as

$$sZ(s) - z_0 = TAT^{-1}Z(s) + TBU(s)$$
$$Y(s) = CT^{-1}Z(s) + DU(s)$$

where $Z(s)$, $Y(s)$, and $U(s)$ denote the vector Laplace transforms of $z(t)$, $y(t)$, and $u(t)$, respectively. Therefore

$$\begin{aligned} Z(s) &= (sI - TAT^{-1})^{-1}TBU(s) + (sI - TAT^{-1})^{-1}z_0 \\ &= [T(sI - A)T^{-1}]^{-1}TBU(s) + [T(sI - A)T^{-1}]^{-1}z_0 \\ &= T(sI - A)^{-1}BU(s) + T(sI - A)^{-1}T^{-1}z_0 \end{aligned}$$

and so it follows that

$$\begin{aligned} Y(s) &= CT^{-1}[T(sI - A)^{-1}BU(s) + T(sI - A)^{-1}T^{-1}z_0] \\ &\quad + DU(s) \\ &= [D + C(sI - A)^{-1}B]U(s) + C(sI - A)^{-1}T^{-1}z_0 \\ &= [D + C(sI - A)^{-1}B]U(s) + C(sI - A)^{-1}x_0 \end{aligned}$$

Hence, Eq. (3) preserves the input-output relationship of dynamic system **S** under any similarity transformation given by $z(t) = Tx(t)$, where $\text{rank}(T) = n$, $T \in \mathbb{R}^{n\times n}$.

To obtain the time-domain solution to Eq. (2), the inverse Laplace transforms of Eq. (4) can be taken to yield

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)Bu(\tau)\,d\tau \tag{5a}$$

and

$$y(t) = C\Phi(t, t_0)x(t_0) + C\int_{t_0}^{t} \Phi(t, \tau)Bu(\tau)\,d\tau + Du(t) \tag{5b}$$

where $\Phi(t, \tau) := e^{A(t-\tau)}$ represents the matrix exponential (22) of A. In addition, given a system modeled by Eq. (2), the following definitions can be made (1).

Definition 1. The dynamic system described in Eq. (2) [or the pair (A, B)] is said to be controllable if, for any given initial state $x(t_0) \in \mathbb{R}^n$, for any finite terminal time $t_1 > t_0$, and for any specified final state $x_f \in \mathbb{R}^n$, there exists a piecewise continuous input $u(t) \in \mathbb{R}^m$ such that the solution of Eq. (2) satisfies $x(t_1) = x_f$. If such an input does not exist, then the system [or the pair (A, B)] is said to be uncontrollable.

A dual to Definition 1 can also be given as shown next.

Definition 2. The dynamic system described in Eq. (2) having initial condition $x(t_0) \in \mathbb{R}^n$ [or the pair (C, A)] is said to be observable if, for any finite time $t_1 > t_0$, the initial state $x(t_0)$ can be determined from measurements of the system input, $u(t) \in \mathbb{R}^m$, and the system output, $y(t) \in \mathbb{R}^r$, for $t \in [t_0, t_1]$. If such a reconstruction of state $x(t_0)$ does not exist, then the system [or the pair (C, A)] is said to be unobservable.

In Eq. (2), one can show (1) that the system is controllable if and only if

$$\operatorname{rank}\,[B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B] = n$$

and that the system is observable if and only if

$$\operatorname{rank}\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

Furthermore, (A, B) is said to be stabilizable if the unstable modes are controllable, and (C, A) is said to be detectable if the unstable modes are observable. It therefore follows that all controllable systems are stabilizable and that all observable systems are detectable. Note, however, that the system in Eq. (2) can be stabilized using a feedback controller with input y and output u if and only if (A, B) is stabilizable and (C, A) is detectable.

Example 1. Consider the two input-two output MIMO system given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{B} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{6a}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{6b}$$

Observe that this system is controllable and observable since

$$\operatorname{rank}[B \quad AB] = 2$$

and

$$\operatorname{rank}\begin{bmatrix} C \\ CA \end{bmatrix} = 2$$

One also can verify that the system is stable (eig(A) = {−1, −2}), and that the transfer function matrix of Eq. (6) is given as

$$\begin{aligned} Y(s) &= [D + C(sI - A)^{-1}B]U(s) \\ &= \begin{bmatrix} \dfrac{2(5s+7)}{(s+1)(s+2)} & \dfrac{2(7s+10)}{(s+1)(s+2)} \\ \dfrac{(11s+31)}{(s+1)(s+2)} & \dfrac{6(3s+8)}{(s+1)(s+2)} \end{bmatrix} U(s) \end{aligned}$$

Note that

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}}_{\overline{D}} = AV$$

and hence

$$\begin{aligned} e^{At} &= I + At + \frac{(At)^2}{2} + \cdots && \text{(7a)} \\ &= V\left[I + V^{-1}AVt + \frac{V^{-1}A^2Vt^2}{2} + \cdots\right]V^{-1} && \text{(7b)} \\ &= V\left[I + V^{-1}AVt + \frac{(V^{-1}AVt)^2}{2} + \cdots\right]V^{-1} && \text{(7c)} \\ &= V\left[I + \overline{D}t + \frac{(\overline{D}t)^2}{2} + \cdots\right]V^{-1} && \text{(7d)} \\ &= Ve^{\overline{D}t}V^{-1} && \text{(7e)} \end{aligned}$$

Let $x(0) := [1 \quad 2]^T$ and define

$$u(t) = [u_1(t) \quad u_2(t)]^T = \begin{cases} [1 \quad 2]^T, & t \geq 0 \\ [0 \quad 0]^T, & t < 0 \end{cases}$$

so that

$$U(s) = \begin{bmatrix} \frac{1}{s} & \frac{2}{s} \end{bmatrix}^T$$

Then from Eqs. (5) and (7),

$$\begin{aligned} x(t) &= \begin{bmatrix} e^{-t} & e^{-t} - e^{-2t} \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &\quad + \int_0^t \begin{bmatrix} e^{-(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ 0 & e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} d\tau \\ &= \begin{bmatrix} 10.5 - 13e^{-t} + 3.5e^{-2t} \\ 5.5 - 3.5e^{-2t} \end{bmatrix}, \; t \geq 0 \end{aligned}$$

and

$$\begin{aligned} y(t) &= \begin{bmatrix} 1 & 3 \\ 5 & 2 \end{bmatrix} x(t) \\ &= \begin{bmatrix} 27 - 13e^{-t} - 7e^{-2t} \\ 63.5 - 65e^{-t} + 10.5e^{-2t} \end{bmatrix}, \; t \geq 0 \end{aligned}$$

Additional information concerning various representations and canonical forms for both continuous- and discrete-time state-space systems can be found, for example, in Refs. 1 and 2.

DYNAMIC INVERSION OF LTI MIMO SYSTEMS

In the general problem of dynamic inversion for LTI MIMO continuous-time systems of the form given in Eq. (2), the goal is to obtain, if possible, a second LTI continuous-time dynamical system that, when cascaded with the original system, produces at its output the input to the original system. Such issues typically arise when investigating filtering and prediction theory, decoupling of multivariable control systems, pursuit-evasion games, and classical sensitivity theory (3).

Define

$$G(s) := D + C(sI - A)^{-1}B$$

to be the $r \times m$ rational transfer function matrix of Eq. (2), and let $\hat{G}(s)$ be a $\hat{r} \times \hat{m}$ (possibly improper) transfer function matrix. The following definitions will be needed.

Definition 3. System $\hat{G}(s)$ is a left inverse (right inverse) of $G(s)$ if $\hat{G}(s)G(s) = I$ ($G(s)\hat{G}(s) = I$).

Definition 4. System $G(s)$ is left invertible (right invertible) if, over the field of rational functions in s, rank$[G(s)] = m$ (rank$[G(s)] = r$).

Note that by Definition 3, it is required that $\hat{m} = r$ and that $\hat{r} = m$. In addition, for the SISO case when $m = r = 1$, left and right invertibility of system $G(s)$ are equivalent.

The following result gives a simple criterion that can be used to determine the right and/or left invertibility of a LTI MIMO continuous-time system in terms of its general matrix parameters (C, A, B, D). In particular, the results presented here differ from the rank condition of the matrix whose elements consist of the Markov parameters D, CB, . . ., $CA^{n-1}B$ (3).

Theorem 1. As in Ref. 4, consider the system given in Eq. (2), and its transfer function matrix $G(s) = D + C(sI - A)^{-1}B$. The following three conditions are equivalent:

1. $G(s)$ has a left inverse.
2. rank$[G(s)] = m$.
3. $$\text{rank}\underbrace{\begin{bmatrix} -A & B & 0 & 0 & \dots & 0 & 0 \\ -C & D & 0 & 0 & \dots & 0 & 0 \\ I & 0 & -A & B & \dots & 0 & 0 \\ 0 & 0 & -C & D & \dots & 0 & 0 \\ 0 & 0 & I & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -A & B \\ 0 & 0 & 0 & 0 & \dots & -C & D \\ 0 & 0 & 0 & 0 & \dots & I & 0 \end{bmatrix}}_{(n+1)\times(n+m)\text{ columns}} = (n+1) \times (n+m)$$

Corollary 1. Condition 3 of Theorem 1 is equivalent to the following condition:

$$\text{rank}\underbrace{\begin{bmatrix} B & AB & \dots & A^{n-1}B & A^nB \\ D & CB & \dots & CA^{n-2}B & CA^{n-1}B \\ 0 & D & \dots & CA^{n-3}B & CA^{n-2}B \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & D & CB \\ 0 & 0 & \dots & 0 & D \end{bmatrix}}_{(n+1)m\text{ columns}} = (n+1)m$$

ROBUST CONTROL

Using the general structure of state-space realizations, the problems of robust control and uncertain linearized systems now can be considered. Such issues are of particular importance since any mathematical model of a physical plant only approximates the actual behavior of the plant. Correspondingly, given a stable LTI system with real parameter perturbations, another area of interest is the real stability radius problem, in which it is desired to determine the largest parametric perturbation that may ocur such that the resultant perturbed system still remains stable.

Parametric Uncertainty

When modeling physical systems, it is generally the case that an exact knowledge of the physical parameters of the system is not known, but that only an approximate knowledge is. For instance, in a dc motor, the inductance of the electric coil windings may vary with time or may be measurable only to a certain degree of precision. In this case, there exists an uncertainty with respect to the physical parameters of the system, called parametric uncertainty. In designing control systems for plants, it is desirable not only that the nominal plant model should be satisfactorily controlled but also that the plant model, when subject to certain parametric uncertainties, should be adequately controlled. If such a requirement can be met, the controller is said to be robust against a certain type of specified parametric uncertainty.

Uncertain Linearized Systems

Although virtually all industrial plants are nonlinear in nature, for controller synthesis purposes, it is common to approximate the behavior of such systems by using LTI models. When doing so, it is assumed that the physical plant can be described by the nonlinear system

$$\dot{x} = f(x, u) \tag{8a}$$

$$y = g(x, u) \tag{8b}$$

where $(x, u, y) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r$ are the plant states, inputs, and outputs, respectively, and $f \in \mathbb{R}^n$ and $g \in \mathbb{R}^r$ are analytic (but not necessarily known) functions. In addition, it is assumed that for some constant input $\overline{u} \in \mathbb{R}^m$, there exists an equilibrium point $\overline{x} \in \mathbb{R}^n$ to Eq. (8) with $\overline{y} := g(\overline{x}, \overline{u})$ such that $f(\overline{x}, \overline{u}) = 0$.

Let δx, δu, and δy denote, respectively, the perturbations about $\overline{x}$, $\overline{u}$, and $\overline{y}$. Then the linearized model of the system about the equilibrium point $\overline{x}$, $\overline{u}$, and $\overline{y}$ is given by

$$\begin{bmatrix} \delta\dot{x} \\ \delta y \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial u} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial u} \end{bmatrix}_{\substack{x=\bar{x}, \\ u=\bar{u}}} \begin{bmatrix} \delta x \\ \delta u \end{bmatrix} + \text{higher order terms}$$

which can be described approximately by

$$\begin{bmatrix} \delta\dot{x} \\ \delta y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \end{bmatrix} + \begin{bmatrix} \Delta A & \Delta B \\ \Delta C & \Delta D \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \end{bmatrix} \tag{9}$$

Eq. (9) implies that the linearized model of the system given by Eq. (8) about $x = \overline{x}$, $u = \overline{u}$, and $y = \overline{y}$, can be described as

$$\delta\dot{x} = (A + \Delta A)\delta x + (B + \Delta B)\delta u \tag{10a}$$

$$\delta y = (C + \Delta C)\delta x + (D + \Delta D)\delta u \tag{10b}$$

where $(C, A, B, D) \in \mathbb{R}^{r\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{r\times m}$ are real matrices representing the nominal linearized model of the system, and $(\Delta C, \Delta A, \Delta B, \Delta D)$ are real matrices representing the uncertainty in the model description. It is to be noted that in view of the approximation made in simplifying Eq. (8) to Eqs. (9)–(10), no a priori structure can necessarily be imposed on the form of the real perturbation matrices ΔA, ΔB, ΔC, ΔD.

Real Stability Radius Problem

Let the complex plane $\mathbb{C}$ be partitioned into two disjoint subsets $\mathbb{C}_g$ and $\mathbb{C}_b$ such that $\mathbb{C} = \mathbb{C}_g \cup \mathbb{C}_b$, where $\mathbb{C}_g$ is a specified open region of $\mathbb{C}^-$; then in this section, the system $\dot{x} = Ax$ will be said to be stable if $\text{eig}(A) \subset \mathbb{C}_g$. Let $\partial\mathbb{C}_g$ denote the boundary of $\mathbb{C}_g$, and let $\mathbb{F}$ be either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$.

Given a plant modeled by

$$\dot{x} = Ax$$

where $A \in \mathbb{R}^{n\times n}$ is real and stable, assume that the plant parameters are subject to uncertainty; that is

$$A \to A + B\Delta C$$

where $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{r\times n}$ are real specified matrices, and $\Delta \in \mathbb{F}^{m\times r}$ is a matrix of uncertain parameters. Then the complex stability radius and the real stability radius are defined as

$$r_{\mathrm{C}}(A, B, C) := \inf\{\overline{\sigma}(\Delta) : \Delta \in \mathbb{C}^{m\times r}, \text{eig}(A + B\Delta C) \not\subset \mathbb{C}_g\}$$

and

$$r_{\mathrm{R}}(A, B, C) := \inf\{\overline{\sigma}(\Delta) : \Delta \in \mathbb{R}^{m\times r}, \text{eig}(A + B\Delta C) \not\subset \mathbb{C}_g\}$$

respectively, where $\overline{\sigma}(*)$ denotes the largest singular value (22) of $(*)$.

It is clear that $r_{\mathrm{R}}(A, B, C) \geq r_{\mathrm{C}}(A, B, C)$. Let $H(s) := C(sI - A)^{-1}B$; in this case, $r_{\mathrm{C}}(A, B, C)$ and $r_{\mathrm{R}}(A, B, C)$ can be determined as follows (5):

$$r_{\mathrm{C}}(A, B, C) = \frac{1}{\sup\limits_{s\in\partial\mathbb{C}_g} \bar{\sigma}[H(s)]} \tag{11a}$$

$$r_{\mathrm{R}}(A, B, C) = \frac{1}{\sup\limits_{s\in\partial\mathbb{C}_g} \inf\limits_{\gamma\in(0,1]} \sigma_2 \begin{bmatrix} \mathrm{Re}[H(s)] & -\gamma\,\mathrm{Im}[H(s)] \\ \frac{1}{\gamma}\mathrm{Im}[H(s)] & \mathrm{Re}[H(s)] \end{bmatrix}} \tag{11b}$$

where $\sigma_2(*)$ denotes the second largest singular value of $(*)$, and $\mathrm{Re}(*)$ and $\mathrm{Im}(*)$ denote, respectively, the real and imaginary parts of $(*)$. Here, $\inf(*)$ is a unimodal function on $(0, 1]$, so that $r_{\mathrm{R}}(A, B, C)$ can be determined effectively.

Example 2. Given

$$A = \begin{bmatrix} -1 & 10^3 & 10^{-3} \\ -1 & -1 & 0 \\ 1 & 1 & -100 \end{bmatrix}$$

$B = I$, and $C = I$, let $\mathbb{C}_g$ be the open left half of the complex plane; then, because $\text{eig}(A) = \{-1 \pm j31.6, -100\}$, where $j = \sqrt{-1}$, A is stable. It may be verified using Eq. (11) that $r_{\mathrm{R}}(A, B, C) = 1.0009$ and that $r_{\mathrm{C}}(A, B, C) = 6.317 \times 10^{-2}$. Hence, in this particular example, the complex stability radius is approximately 16 times smaller than the real stability radius. As well, the smallest real perturbation matrix that destabilizes this system is given by

$$\Delta = \begin{bmatrix} 9.9980 \times 10^{-4} & 9.9879 \times 10^{-7} & 1.0009 \times 10^{-5} \\ 1.0009 & 9.9989 \times 10^{-4} & 1.0020 \times 10^{-2} \\ 1.0030 \times 10^{-4} & 1.0020 \times 10^{-7} & 1.0041 \times 10^{-6} \end{bmatrix}$$

THE SERVOMECHANISM PROBLEM

In the servomechanism problem, it is desired to design a controller for a system so that closed loop stability is maintained and so that asymptotic tracking and rejection occur, respectively, for a given class of reference and disturbance input signals. This section studies controller synthesis methods for the servomechanism problem when the plant is subject to uncertainty. In particular, it is desired to design a controller so that asymptotic reference tracking and disturbance regulation occur, in spite of the fact that the plant dynamics may be perturbed by arbitrary large amounts, subject only to the condition that the resultant closed loop perturbed system also remains stable. This problem is called the robust servomechanism problem.

Preliminary Definitions and Results

The plant to be controlled is assumed to be described by the following linear time-invariant model:

$$\dot{x} = Ax + Bu + E\omega \tag{12a}$$

$$y = Cx + Du + F\omega \tag{12b}$$

$$y_{\mathrm{m}} = C_{\mathrm{m}}x + D_{\mathrm{m}}u + F_{\mathrm{m}}\omega \tag{12c}$$

$$e = y_{\mathrm{r}} - y \tag{12d}$$

where $x \in \mathbb{R}^n$ are the states, $u \in \mathbb{R}^m$ are the inputs that can be manipulated, $y \in \mathbb{R}^r$ are the plant outputs that are to be regulated, and $y_{\mathrm{m}} \in \mathbb{R}^{r_{\mathrm{m}}}$ are the plant outputs that can be measured. Here $\omega \in \mathbb{R}^{\Omega}$ corresponds to the disturbances in the system, which in general cannot necessarily be measured, and $e \in \mathbb{R}^r$ is the error in the system, which is the difference between the reference input signal y_{r}, in which it is desired that the output y should track, and the plant output y.

It is assumed that the disturbances ω arise from the following class of systems:

$$\dot{\eta}_1 = \mathscr{A}_1\eta_1, \qquad \omega = \mathscr{C}_1\eta_1; \qquad \eta_1 \in \mathbb{R}^{n_1} \tag{13a}$$

and that the reference input signals y_{r} arise from the following class of systems:

$$\dot{\eta}_2 = \mathscr{A}_2\eta_2, \qquad \rho = \mathscr{C}_2\eta_2, \qquad y_{\mathrm{r}} = G\rho; \qquad \eta_2 \in \mathbb{R}^{n_2} \tag{13b}$$

For nontriviality, assume that $\text{eig}(\mathscr{A}_1) \subset \mathbb{C}^+ \cup \mathbb{C}^0$ and that $\text{eig}(\mathscr{A}_2) \subset \mathbb{C}^+ \cup \mathbb{C}^0$. It is also assumed with no loss of generality that $(\mathscr{C}_1, \mathscr{A}_1)$ and $(\mathscr{C}_2, \mathscr{A}_2)$ are observable, and that

$$\text{rank}\begin{bmatrix} E \\ F \end{bmatrix} = \text{rank}\,\mathscr{C}_1 = \dim(\omega)$$

and

$$\text{rank}\, G = \text{rank}\, \mathscr{C}_2 = \dim(\rho)$$

This family of signals includes most classes of signals that occur in application problems (e.g., constant, polynomial, sinusoidal, polynomial-sinusoidal). The following definitions will be used in subsequent discussions.

Definition 5. Given the systems represented by Eq. (13), let $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ be the zeros of the least common multiple of the minimal polynomial of $\mathscr{A}_1$ and the minimal polynomial of $\mathscr{A}_2$ (multiplicities repeated), and call

$$\Lambda := \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$$

the disturbance/tracking poles of Eq. (13).

Definition 6. Consider the system

$$\dot{x} = Ax + Bu; \qquad u \in \mathbb{R}^m, \qquad y \in \mathbb{R}^r, \qquad x \in \mathbb{R}^n \tag{14a}$$

$$y = Cx + Du \tag{14b}$$

Then $\lambda \in \mathbb{C}$ is said to be a transmission zero (10) of (C, A, B, D) if

$$\text{rank} \begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix} < n + \min(r, m)$$

In particular, the transmission zeros are the zeros (multiplicities included) of the greatest common divisor of all $[n + \min(r, m)] \times [n + \min(r, m)]$ minors of

$$\begin{bmatrix} A - \lambda I & B \\ C & D \end{bmatrix}$$

Definition 7. Given the system (C, A, B, D) in Eq. (14), assume that one or more of the transmission zeros of (C, A, B, D) are contained in the closed right half complex plane $\mathbb{C}^+ \cup \mathbb{C}^0$; then (C, A, B, D) is said to be a nonminimum phase system. If (C, A, B, D) is not a nonminimum phase system, then it is said to be a minimum phase system.

The robust servomechanism problem for Eq. (12) consists of finding a linear time-invariant controller that has inputs y_m, y_r, and outputs u for the plant so that:

1. The resultant closed loop system is stable.
2. Asymptotic tracking occurs; that is,

$$\lim_{t\to\infty} e(t) = 0, \quad \forall x(0) \in \mathbb{R}^n, \quad \forall \eta_1(0) \in \mathbb{R}^{n_1}, \quad \forall \eta_2(0) \in \mathbb{R}^{n_2}$$

   for all controller initial conditions.
3. For any arbitrary perturbations in the plant model given by Eq. (12) (including, for example, changes in model order, plant parameters, or plant dynamics) that do not cause the resultant perturbed closed loop system to become unstable, condition 2 holds.

Main Results

The following results are obtained concerning the existence of a solution to the robust servomechanism problem (6,7).

Theorem 2. There exists a solution to the robust servomechanism problem for Eq. (12) if and only if the following conditions all are satisfied:

1. (C_m, A, B) is stabilizable and detectable.
2. $m \geq r$.
3. The transmission zeros of (C, A, B, D) exclude the disturbance/tracking poles λ_i, $i \in \{1, 2, \ldots, p\}$.
4. $y \subset y_m$ (i.e., the outputs y are measurable).

Remark 2. Conditions 2 and 3 in Theorem 2 are equivalent to the condition that

$$\text{rank} \begin{bmatrix} A - \lambda_i I & B \\ C & D \end{bmatrix} = n + r, \qquad i \in \{1, 2, \ldots, p\} \tag{15}$$

The following definitions of a stabilizing compensator and a servocompensator are required for the ensuing development.

Definition 8. Given the stabilizable and detectable system (C_m, A, B, D_m) obtained from Eq. (12), a linear time-invariant stabilizing compensator

$$\dot{\xi} = \Lambda_1 \xi + \Lambda_2 y_m$$
$$u = K_1 \xi + K_2 y_m$$

is defined to be a controller that stabilizes the resultant closed loop system such that desired transient behavior occurs.

This compensator is not a unique device, and may be designed by using a number of different techniques.

Definition 9. Given the disturbance/tracking poles λ_i, $i \in \{1, 2, \ldots, p\}$, the matrix $\mathscr{C} \in \mathbb{R}^{p\times p}$ and the vector $\gamma \in \mathbb{R}^p$ are defined by

$$\mathscr{C} := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -\delta_1 & -\delta_2 & -\delta_3 & \cdots & -\delta_p \end{bmatrix}, \quad \gamma := \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{16}$$

where the coefficients δ_i are given by the coefficients of the polynomial $\prod_{i=1}^{p} (\lambda - \lambda_i)$; that is,

$$\lambda^p + \delta_p \lambda^{p-1} + \cdots + \delta_2 \lambda + \delta_1 := \prod_{i=1}^{p} (\lambda - \lambda_i)$$

The following compensator, called a servocompensator, is of primary importance in the design of controllers to solve the robust servomechanism problem (6).

Definition 10. Consider the class of disturbance/reference signals given by Eq. (13) and consider the system given by Eq. (12); then a servocompensator for Eq. (12) is a controller with input $e \in \mathbb{R}^r$ and output $\eta \in \mathbb{R}^{rp}$ given by

$$\dot{\eta} = \mathscr{C}^* \eta + \mathscr{B}^* e \tag{17a}$$

where

$$\mathscr{C}^* := \text{block diag}(\underbrace{\mathscr{C}, \mathscr{C}, \ldots, \mathscr{C}}_{r}) \tag{17b}$$

$$\mathscr{B}^* := \text{block diag}(\underbrace{\gamma, \gamma, \ldots, \gamma}_{r}) \tag{17c}$$

and where $\mathscr{C}$ and γ are given by Eq. (16).

The servocompensator is unique within the class of coordinate transformations and nonsingular input transformations. In addition, given the servocompensator defined in Eq. (17), let $\mathscr{D} \in \mathbb{R}^{r \times rp}$ be defined by

$$\mathscr{D} := \text{block diag}(\underbrace{\alpha, \alpha, \ldots, \alpha}_{r})$$

where $\alpha \in \mathbb{R}^{1 \times p}$ is given by

$$\alpha := [1 \quad 0 \quad 0 \quad \cdots \quad 0]$$

The servocompensator has the following properties.

Lemma 1. As in Ref. 9, given the plant modeled by Eq. (12), assume that the existence conditions of Theorem 2 all hold; then:

1. The system

$$\left\{ \begin{bmatrix} C_m & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} A & 0 \\ -\mathscr{B}^* C & \mathscr{C}^* \end{bmatrix}, \begin{bmatrix} B \\ -\mathscr{B}^* D \end{bmatrix} \right\}$$

is stabilizable and detectable and has centralized fixed modes (8) (i.e., those modes of the system that are not both simultaneously controllable and observable) equal to the centralized fixed modes of (C_m, A, B, D_m).

2. The transmission zeros of

$$\left\{ [0 \quad \mathscr{D}], \begin{bmatrix} A & 0 \\ -\mathscr{B}^* C & \mathscr{C}^* \end{bmatrix}, \begin{bmatrix} B \\ -\mathscr{B}^* D \end{bmatrix} \right\}$$

are equal to the transmission zeros of (C, A, B, D).

Robust Servomechanism Controller

Consider the system given by Eq. (12) and assume that the existence conditions of Theorem 2 hold; then the following LTI controller solves the robust servomechanism problem for Eq. (12) (7):

$$u = \xi + K\eta \tag{18}$$

where $\eta \in \mathbb{R}^{rp}$ is the output of the servocompensator given by Eq. (17) and ξ is the output of a stabilizing compensator $\mathscr{S}$ with inputs y_m, y_r, η, and u. $\mathscr{S}$ and K are constructed to stabilize and give desired behavior to the following stabilizable and detectable system:

$$\begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A & 0 \\ -\mathscr{B}^* C & \mathscr{C}^* \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix} + \begin{bmatrix} B \\ -\mathscr{B}^* D \end{bmatrix} u$$

$$\tilde{y}_m = \begin{bmatrix} C_m & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix} + \begin{bmatrix} D_m \\ 0 \\ I \end{bmatrix} u$$

From Lemma 1, the centralized fixed modes (if any) of

$$\left\{ \begin{bmatrix} C_m & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} A & 0 \\ -\mathscr{B}^* C & \mathscr{C}^* \end{bmatrix}, \begin{bmatrix} B \\ -\mathscr{B}^* D \end{bmatrix} \right\}$$

are equal to the centralized fixed modes of (C_m, A, B, D_m). It is to be noted, however, that the controller given by Eq. (18) always has order $\geq rp$.

Some properties of the robust servomechanism controller (6) represented by Eq. (18) follow:

1. In the robust servomechanism problem, it is only required to know the disturbance/tracking poles $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$; that is, it is not necessary to know E or F of Eq. (12) nor $\mathscr{A}_1$, $\mathscr{A}_2$, $\mathscr{C}_1$, $\mathscr{C}_2$, or G of Eq. (13).
2. A controller exists generically (11) for almost all plants described by Eq. (12), provided that: (a) $m \geq r$, and (b) the outputs y can be measured. If either (a) or (b) fails to hold, there is no solution to the robust servomechanism problem.

Example 3. Consider the unstable two input-two output headbox model taken from Ref. 9, where

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0.395 & 0.01145 \\ -0.011 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0.03362 & 1.038 \\ 0.000966 & 0 \end{bmatrix}}_{B} u + E\omega$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + F\omega$$

with $y_m = y$, and assume that it is desired to solve the robust servomechanism problem for constant reference and constant disturbance input signals. In this case, the disturbance/tracking poles are equal to {0}, and one can verify that the existence conditions given in Theorem 2 hold. In addition, since the servocompensator (17) is given as

$$\dot{\eta} = y_r - y$$

it follows that the augmented system is

$$\begin{bmatrix}\dot{x}\\ \dot{\eta}\end{bmatrix} = \underbrace{\begin{bmatrix}A & 0\\ -C & 0\end{bmatrix}}_{\overline{A}}\begin{bmatrix}x\\ \eta\end{bmatrix} + \underbrace{\begin{bmatrix}B\\ 0\end{bmatrix}}_{\overline{B}}u + \begin{bmatrix}E\\ -F\end{bmatrix}\omega + \begin{bmatrix}0\\ I\end{bmatrix}y_{\mathrm{r}}$$

$$\begin{bmatrix}y\\ \eta\end{bmatrix} = \underbrace{\begin{bmatrix}C & 0\\ 0 & I\end{bmatrix}}_{\overline{C}}\begin{bmatrix}x\\ \eta\end{bmatrix} + \begin{bmatrix}F\\ 0\end{bmatrix}\omega$$

Observe now that $(\overline{A}, \overline{B})$ is controllable and that $\overline{C} = I$; hence, pole placement can be used to design the feedback control law

$$u = \begin{bmatrix} 11.387 & -7246.4 & 0 & 12422\\ -3.6395 & 234.69 & 1.9268 & -402.35\end{bmatrix}\begin{bmatrix}y\\ \eta\end{bmatrix}$$

which yields closed loop eigenvalues of $\{-1, -2, -3, -4\}$. Moreover, for the case when

$$E = \begin{bmatrix}1\\ 1\end{bmatrix}$$
$$F = \begin{bmatrix}2\\ 5\end{bmatrix}$$
$$x(0) = 0$$
$$\omega(t) = 1$$

and

$$y_{\mathrm{r}}(t) = \begin{bmatrix}1\\ -1\end{bmatrix}$$

one can see that a desirable transient response as well as tracking and regulation also occur as shown in Fig. 2.

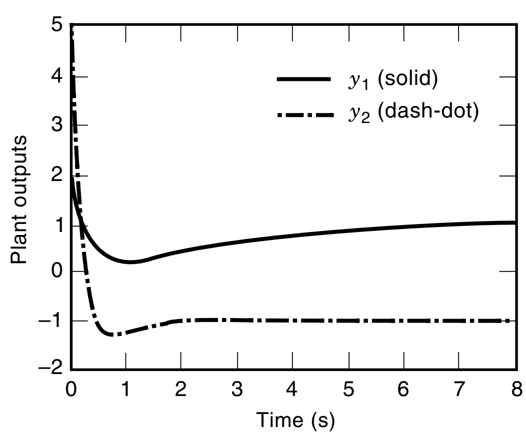

Figure 2. A sample output response of the final compensated headbox given in Example 3.

ADAPTIVE SWITCHING CONTROL

During the past several years, there has been a considerable amount of interest and effort made toward developing controller design methods that endeavor to use as little a priori plant information as possible (12–21). The motivation for this interest stems from the fact that it is generally difficult and often impossible to obtain an accurate model representation of an actual industrial plant.

Currently, one method to deal effectively with the specific problem of parametric plant uncertainty is adaptive control (23–25). However, while the controllers employed in such schemes typically are nonlinear and time-varying, and consist of a compensator augmented with a tuning mechanism that adjusts the compensator gains, important a priori plant information still is required. For example, in conventional model reference adaptive control of a SISO system, the four classical assumptions typically made are that (23,24)

1. the plant is minimum phase;
2. an upper bound on the plant order exists and is known;
3. the relative degree is known; and
4. the sign of the high-frequency gain is known.

Although recent developments have been able to remove condition 4 (26,27) and to weaken conditions 2 (28) and 3 (29,30), specific plant information [e.g., any plant zeros that lie in the open right half complex plane must be known to lie in a finite set (31)] still is needed.

This section examines one particular robust multivariable switching approach that has been successfully implemented (16–18) using very little a priori system information. The controller proposed here is a self-tuning switching controller, which has the property that, after a finite time, the controller stops switching and simplifies to a LTI controller.

Preliminary Definitions and Results

Plant Model. Let each element

$$P_i := (A_i, B_i, C_i, D_i, E_i, F_i),\ i \in \{1, 2, \ldots, s\},\ s \in \mathbb{N}$$

belonging to the finite set of possible plants to be controlled

$$\mathbf{P} := \bigcup_{i=1}^{s} P_i$$

be of the finite dimensional form

$$\dot{x} = A_i x + B_i u + E_i\omega$$
$$y = C_i x + D_i u + F_i\omega$$
$$e := y_{\mathrm{r}} - y$$

where $x \in \mathbb{R}^{n_i}$ is the state, $u \in \mathbb{R}^m$ is the control input that can be manipulated, $y \in \mathbb{R}^r$ is the plant output that is to be regulated, $\omega \in \mathbb{R}^q$ is the disturbance, and $e \in \mathbb{R}^r$ is the difference between the specified reference input y_{r} and the plant output y. In the discussion that follows, we do not assume that n_i, A_i, B_i, C_i, D_i, E_i, or F_i are known necessarily.

Class of Candidate Controllers. Let each candidate feedback controller be of the finite dimensional form given by

$$\mathscr{K}_i : \begin{cases}\dot{\eta} & = G_i\eta + H_i y + J_i y_{\mathrm{r}}\\ u & = K_i\eta + L_i y + M_i y_{\mathrm{r}}\end{cases},\ i \in \{1, 2, \ldots, s\}$$

where $s \in \mathbb{N}$, $\eta \in \mathbb{R}^{g_i}$, $G_i \in \mathbb{R}^{g_i \times g_i}$, $H_i \in \mathbb{R}^{g_i \times r}$, $J_i \in \mathbb{R}^{g_i \times r}$, $K_i \in \mathbb{R}^{m \times g_i}$, $L_i \in \mathbb{R}^{m \times r}$, and $M_i \in \mathbb{R}^{m \times r}$.

Remark 3. Given $D \in \mathbb{R}^{r \times m}$, then for almost all (11) $L \in \mathbb{R}^{m \times r}$, $(I - DL) \in \mathbb{R}^{r \times r}$ is invertible. Hence, given a fixed matrix D, then $(I - DL)$ is invertible for generic L.

The following definition also will be needed.

Definition 11. A function $f : \mathbb{N} \to \mathbb{R}^+$ is said to be a bounding function ($f \in$ BF) if it is strictly increasing and if, for all constants $(c_0, c_1, c_2) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$,

$$\frac{f(i)}{c_0 + c_1(i-1) + c_2 \sum_{j=1}^{i-1} f(j)} \to \infty$$

as $i \to \infty$.

Proposition 1. There exists a bounding function (e.g., $f(i) = i\exp(i^2)$).

Main Results

For the situation when $y_r(t)$ and $\omega(t)$ are bounded piecewise continuous signals, label Controller 1 as

$$\begin{aligned} \dot{e}_f &= -\lambda e_f + \lambda e,\ \lambda \in \mathbb{R}^+ \\ \dot{\eta}(t) &= G(t)\eta(t) + H(t)y(t) + J(t)y_r(t) \\ \eta(t_k^+) &\equiv 0 \\ u(t) &= K(t)\eta(t) + L(t)y(t) + M(t)y_r(t) \end{aligned}$$

where $k \in \{1, 2, 3, \ldots\}$, $i := [(k-1) \bmod s] + 1$,

$$(G(t), H(t), J(t), K(t), L(t), M(t)) := (G_i, H_i, J_i, K_i, L_i, M_i), \quad t \in (t_k, t_{k+1}]$$

$t_1 := 0$, and where, for each $k \geq 2$ such that $t_{k-1} \neq \infty$, the switching time t_k is defined by

$$t_k := \begin{cases} \min t \text{ such that} & \\ \text{i) } t > t_{k-1}, \text{ and} & \text{if this minimum exists} \\ \text{ii) } \|[\eta(t)^T e_f(t)^T]^T\| = f(k-1) & \\ \infty & \text{otherwise} \end{cases}$$

with $f \in$ BF. In addition, let Assumption 1 be the following:

1. $\|\eta(0)\| < f(1)$;
2. $\|e_f(0)\| < f(1)$;
3. for each plant P_i and for each corresponding applied Controller $\mathcal{K}_i$, $i \in \{1, 2, \ldots, s\}$, the closed loop system is stable and controller parameters (G_i, H_i, J_i, K_i, L_i, M_i) provide acceptable reference tracking/disturbance rejection when the plant P_i is subject to bounded piecewise constant reference and disturbance inputs;
4. for each plant P_i, (C_i, A_i) is detectable; and
5. for each $i, j \in \{1, 2, \ldots, s\}$, $(I - D_iL_j)$ is invertible (see Remark 3).

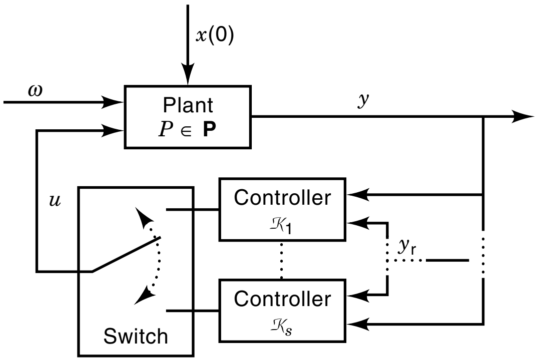

Figure 3. A schematic block diagram of Controller 1.

The switching mechanism described by Controller 1 is schematically depicted in Fig. 3. In Controller 1, norm bounds on $\eta(t)$ and $e_f(t)$ are used in an attempt to detect closed loop instability, which might be caused if Controller $\mathcal{K}_i$ is applied to plant P_j, $i \neq j$. If this upper bound is met at any time during the tuning process, then a controller switch occurs, and η is reset to zero immediately following this switch. This reset action is performed because all candidate feedback controllers need not necessarily be of the same order and because past experimental investigations (16) have indicated that reduced tuning transient responses generally can be attained via such a scheme. However, for the case when all candidate controllers have the same order, $\eta(t_k^+)$ need not necessarily be reset to zero after each switch; one can choose to continue to form $\eta(t)$ using the set of piecewise LTI systems given by (G_i, H_i, J_i) with $\eta(t_k^+) = \eta(t_k)$.

The following result can now be obtained (18).

Theorem 3. Consider a plant $P \in \mathbf{P}$ with Controller 1 applied at time $t = 0$; then for every $f \in$ BF and $\lambda \in \mathbb{R}^+$, for every bounded piecewise continuous reference and disturbance signal, and for every initial condition $\tilde{z}(0) := [x(0)^T\ \eta(0)^T\ e_f(0)^T]^T$ for which Assumption 1 holds, the closed loop system has the properties that:

1. there exist a finite time $t_{ss} \geq 0$ and constant matrices (G_{ss}, H_{ss}, J_{ss}, K_{ss}, L_{ss}, M_{ss}) such that ($G(t)$, $H(t)$, $J(t)$, $K(t)$, $L(t)$, $M(t)$) = (G_{ss}, H_{ss}, J_{ss}, K_{ss}, L_{ss}, M_{ss}) for all $t \geq t_{ss}$;
2. the controller states $\eta \in \mathcal{L}_\infty$, the plant states $x \in \mathcal{L}_\infty$, and the filtered error signal states $e_f \in \mathcal{L}_\infty$; and
3. if the reference and disturbance inputs are constant signals, then for almost all controller parameters (G_{ss}, H_{ss}, K_{ss}, L_{ss}), asymptotic error regulation occurs.

In Theorem 3, the class of reference and disturbance signals potentially allowable for the servomechanism controller design (6) of $\mathcal{K}_i$ and the implementation of Controller 1 actually is larger than the family of constant signals provided that $y_r \in \mathcal{L}_\infty$ and $\omega \in \mathcal{L}_\infty$. For instance, the class of signals given by Eq. (13) may be allowed. Moreover, Theorem 3 clearly will also hold even if the finite number of candidate controllers is greater than or equal to the number of possible plants.

In addition, in Theorem 3, the requirement that $y_r(t)$ and $\omega(t)$ be bounded piecewise continuous functions and the restriction that switching cannot occur infinitely fast guarantees the existence and uniqueness of a solution to the set of closed loop differential equations. Furthermore, without any loss of properties 1–3 given in Theorem 3, filtered error signal $e_f(t)$ could also have been defined as

$$\begin{aligned} \dot{x}_f &= A_f x_f + B_f e \\ e_f &= C_f x_f \end{aligned}$$

where $x_f \in \mathbb{R}^r$, $\mathrm{eig}(A_f) \subset \mathbb{C}^-$, and C_f and B_f are both invertible.

In fact, for the general situation when plant P_i is described by

$$\begin{aligned} \dot{x} &= A_i x + B_i u + E_i \omega + N_i \mu_1 \\ y &= C_i x + D_i u + F_i \omega + Q_i \mu_2 \end{aligned}$$

properties 1 and 2 of Theorem 3 will also hold for all bounded piecewise continuous noise signals $(\mu_1, \mu_2) \in \mathbb{R}^{n_n} \times \mathbb{R}^{q_n}$ with $(N_i, Q_i) \in \mathbb{R}^{n_i \times n_n} \times \mathbb{R}^{r \times q_n}$. This follows because the closed loop system with Controller $\mathscr{K}_i$ applied may be expressed as

$$\begin{bmatrix} \dot{x} \\ \dot{\eta} \\ \dot{e}_f \end{bmatrix} = A_{cl} \underbrace{\begin{bmatrix} x \\ \eta \\ e_f \end{bmatrix}}_{\tilde{z}} + B_{cl} \begin{bmatrix} y_r \\ \omega \\ \mu_1 \\ \mu_2 \end{bmatrix}$$

where

$$\tilde{I} := (I - D_i L_i)^{-1}$$

$$A_{cl} := \begin{bmatrix} A_i + B_i L_i \tilde{I} C_i & B_i K_i + B_i L_i \tilde{I} D_i K_i & 0 \\ H_i \tilde{I} C_i & G_i + H_i \tilde{I} D_i K_i & 0 \\ -\lambda \tilde{I} C_i & -\lambda \tilde{I} D_i K_i & -\lambda I \end{bmatrix}$$

and $B_{cl} :=$

$$\begin{bmatrix} B_i M_i + B_i L_i \tilde{I} D_i M_i & E_i + B_i L_i \tilde{I} F_i & N_i & B_i L_i \tilde{I} Q_i \\ J_i + H_i \tilde{I} D_i M_i & H_i \tilde{I} F_i & 0 & H_i \tilde{I} Q_i \\ \lambda I - \lambda \tilde{I} D_i M_i & -\lambda \tilde{I} F_i & 0 & -\lambda \tilde{I} Q_i \end{bmatrix}$$

Moreover, if $\|[\mu_1^T\ \mu_2^T]^T\| \to 0$, property 3 of Theorem 3 will once again be recovered.

Finally, it should be noted that in contrast with conventional parameter-based methods utilized in adaptive control, the nonparametric approach of Controller 1 possesses the desirable property of being very robust to large plant uncertainties. Unfortunately, however, one particular disadvantage of this scheme is the potential closed loop susceptibility to substantial output transient responses.

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