GYRATORS

Definition

A *gyrator* is a nonreciprocal electrical network. It is capable of transforming signals or energy represented in terms of one electrical quantity, such as voltage or magnetic field, to another electrical quantity that may be of similar type or of a complementary type, such as current or electric field. Such networks are quite useful in electronic systems, since one often wishes to design systems with a limited set of component types or with restrictions regarding certain physical parameters.

Historical Usage

Tellegen first proposed the idea of a gyrator in his original work in 1948 (1). In this paper he explained that resistors, capacitors, inductors, and ideal transformers were the four basic circuit building blocks. However, these elements are all reciprocal and could, therefore, only be expected to go into the creation of reciprocal networks. *Reciprocal networks* are those networks whose impedance (or admittance) matrices are symmetrical. In order to realize nonreciprocal networks, one would need a nonreciprocal building block. Tellegen proposed such a network, calling it a gyrator. This name was given because the equations produced for an electrical network with a gyrator were identical to those of a mechanical gyrostatic network. As time went on other researchers (2–5) picked up on the idea and began to look for circuit realizations for this abstract functional block. In addition, over the years other systems, such as microwave circulators, have been recognized as being analogous to gyrators, which helps in the understanding of these systems.

THE BASICS OF TWO-PORT GYRATORS

Introduction to Two-Ports and Passivity

A gyrator is a special type of electrical two-port network. Electrical two-port networks are any circuits where one can identify two ports, or simply, two pairs of nodes to which one might consider the connection of two pairs of wires. One of the nodes at each port may be in common—for example, ground may be common to both ports. Additionally, a bipolar transistor may be considered a two-port, where port 1 is the base-emitter node pair, and port 2 is the collector-emitter node pair.

Except for trivial cases, every two-port possesses a mathematical description relating the port voltages, V_1 and V_2, and associated port currents, I_1 and I_2. Figure 1 shows the standard reference labeling for the voltages and currents of a two-port. Notice that the port currents are defined as flowing into the + voltage reference for each port. These sign conventions make the definition of power regarding a two-port more precise. Specifically, the power as a function of time, $P(t)$, delivered to a two-port is given by $P(t) = V_1(t)I_1(t) + V_2(t)I_2(t)$, which is analogous to the definition in a one-port—that is, a two terminal element. A two-port network is *lossless* when the average power, P(t), delivered to the network is zero. A network is called *passive* if the average power delivered to the network is positive. Active networks are those networks where P(t) is negative on average. In general, the power delivered to a network can be positive or negative instantaneously, regardless of its passivity. For example, a capacitor in a resonant circuit alternately sinks and sources power instantaneously, despite its lossless average power consumption. Power can be defined just as easily in the frequency domain, again in a way analogous to one-ports. The Fourier transform of power, $P(\omega)$, as a function of frequency is given by, $P(\omega) = V_1(\omega)I_1(\omega)^* + V_2(\omega)I_2(\omega)^*$, where the * indicates complex conjugation.

The general description for a two-port is given by a relation between its port voltages and currents. One such description is the so called y-parameter model given by,

$$\begin{vmatrix} I_1 \\ I_2 \end{vmatrix} = \begin{vmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{vmatrix} \begin{vmatrix} V_1 \\ V_2 \end{vmatrix} \Rightarrow \boldsymbol{I} = Y\boldsymbol{V} \tag{1}$$

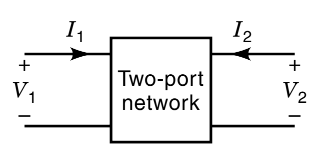

Figure 1. Basic two-port model.

Equation (1) defines the port current and port voltage vectors, $\boldsymbol{I}$ and $\boldsymbol{V}$, respectively, in addition to the y-parameter matrix, Y. By inverting this vector equation, one relates the port voltages to the port currents with z-parameters. Specifically,

$$\begin{vmatrix} V_1 \\ V_2 \end{vmatrix} = \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix} \begin{vmatrix} I_1 \\ I_2 \end{vmatrix} \Rightarrow \boldsymbol{V} = Z\boldsymbol{I} \tag{2}$$

where Z is the z-parameter matrix for the two-port. Using models of the type shown in Eqs. (1) and (2), two-ports can be compared by their two-port parameters—for example, y-parameters. Two-ports characterized by symmetric y-parameter matrices (equivalently z-parameter matrices) are called *reciprocal.* Networks possessing this reciprocity are not necessarily lossless, but can always be realized with reciprocal physical elements. Note that one-ports (two terminal elements) are always reciprocal. The concept of a two-port can be extended in an obvious way to N-ports by considering the voltage-current relationship measured using N pairs of terminals or, equivalently, N-ports.

Mathematical Two-Port Definition of a Gyrator

In the context of this discussion, a gyrator is simply a special case of a linear two-port. While there are many possible two-port descriptions, the most common way of writing the basic equations relating the port parameters in a gyrator is as follows:

$$I_2 = gV_1; \quad I_1 = -gV_2 \tag{3}$$

Using these equations it is simple to write the y-parameter two-port description for a gyrator as,

$$\begin{vmatrix} I_1 \\ I_2 \end{vmatrix} = \begin{vmatrix} 0 & -g \\ g & 0 \end{vmatrix} \begin{vmatrix} V_1 \\ V_2 \end{vmatrix} \Rightarrow \boldsymbol{I} = Y\boldsymbol{V} \tag{4}$$

This suggests that a gyrator can be implemented with voltage controlled current sources, having gains of g and $-g$, respectively. By inverting the relations in Eq. (3), one obtains a gyrator formulation based upon current controlled voltage sources. Specifically,

$$\boldsymbol{V} = \begin{vmatrix} V_1 \\ V_2 \end{vmatrix} = \begin{vmatrix} 0 & -r \\ r & 0 \end{vmatrix} \begin{vmatrix} I_1 \\ I_2 \end{vmatrix} = Z\boldsymbol{I}; \quad r = 1/g \tag{5}$$

where the z-parameter matrix, Z, is just the inverse of the y-parameter matrix, Y. Since it is most convenient to realize practical voltage controlled current source networks, as opposed to current controlled voltage source networks, the formulation in Eq. (4) is generally preferred. For theoretical purposes, of course, both formulations are useful. Using Eq. (4) it is simple to show that a gyrator is a lossless electrical network. Specifically,

$$\begin{aligned} P = \boldsymbol{V}^T\boldsymbol{I} = \begin{vmatrix} V_1 & V_2 \end{vmatrix} \begin{vmatrix} I_1 \\ I_2 \end{vmatrix} = \begin{vmatrix} V_1 & V_2 \end{vmatrix} \begin{vmatrix} 0 & -g \\ g & 0 \end{vmatrix} \begin{vmatrix} V_1 \\ V_2 \end{vmatrix} \\ = V_2V_1 - V_1V_2 = 0 \end{aligned} \tag{6}$$

The fact that gyrators are, in theory, lossless makes them attractive in filter synthesis, and this will become clear later.

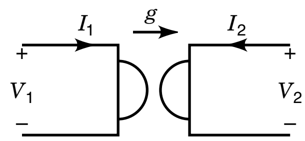

Figure 2. Electrical circuit symbol for a gyrator.

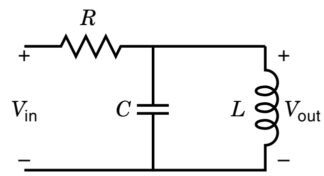

Figure 4. Bandpass filter.

Properties of Gyrators

There are several properties which gyrators possess that make these circuits interesting for use in electronics. A first property is that these two-ports are not reciprocal networks, since their y-parameter matrices are not symmetric. In fact, these matrices are skew symmetric. It is well known to circuit theorists that nonreciprocal networks cannot be realized with only passive components—that is resistors, capacitors, and inductors. This means that gyrators are strictly active networks that must, therefore, be realized with active components, such as transistors or operational amplifiers. Two-port gyrators have been given their own circuit symbol which is shown in Fig. 2. The gyration constant, g is built into the symbol.

Perhaps the most important property of a gyrator is its ability to transform admittances into impedances. Specifically, when an admittance is connected to one port of a gyrator, the impedance looking into the other port is exactly a scaled version of that admittance. The derivation can be accomplished with the help of Fig. 3, where Z_{load} is the impedance attached to port 2, Y_{load} is its reciprocal—that is, the admittance attached to port 2—and Z_{input} is the impedance seen looking into port 1. We have,

$$Z_{\text{load}} = \frac{V_2}{-I_2}; \quad Z_{\text{input}} = \frac{V_1}{I_1} = \frac{I_2/g}{-gV_2} = \frac{1}{g^2}\frac{1}{Z_{\text{load}}} = \frac{1}{g^2}Y_{\text{load}} \tag{7}$$

The gyration constant, g, determines the scale factor, but the nature of the input impedance is determined by the admittance attached to port 2. Therefore, if a capacitor is attached to port 2, then we have,

$$Z_{\text{input}} = \frac{1}{g^2}sC = sL_{\text{eq}}; \quad L_{\text{eq}} = \frac{C}{g^2} \tag{8}$$

This simple relation explains the vast majority of the gyrator's popularity in electronic design. It shows that a capacitor can be used to replace an inductor in a circuit with the help of a gyrator. Since inductors are rarely desirable in electronic circuits operating below about 1 GHz, this idea is quite appealing. Capacitors and gyrators are conveniently realized within integrated circuits.

Filter synthesis based on the inductor simulation already described is usually done by starting with an *RLC* (resistor, inductor, capacitor) prototype, and replacing the inductors with capacitor/gyrator combinations. Consider the following example of a very simple second-order bandpass filter shown in Fig. 4. After replacing the inductor with a gyrator/capacitor combination, the filter is realized solely using *RC* (resistor, capacitor) passive elements, as shown in Fig. 5. Some original related work appears in Refs. 6 and 7. Furthermore, the filter can be tuned electronically if the gyration constant can be varied electronically. An electronically tunable filter using gyrators will be shown later.

A byproduct of the property under discussion is that series and parallel circuits may be interchanged with the help of a gyrator. Suppose one port, say port 2, of a gyrator is loaded with a parallel combination of elements. The admittance of this combination is the sum of the admittances of each of the elements. At the other port, port 1, the input impedance will be a scaled version of this admittance; hence, a sum of impedances. Since the composite input impedance seen at port 1 is given by a sum of impedances, it must be equivalent to a series combination of elements. Therefore, the gyrator converts a parallel network into a series network. Using similar logic, it becomes clear that a series network connected to port 2 will be reflected as a parallel network looking into port 1. These results are summarized next.

$$\begin{aligned} Y_{\text{load}} &= \sum_{k=1}^{N} Y_{\text{k}} \Rightarrow Z_{\text{input}} = \frac{Y_{\text{load}}}{g^2} = \sum_{k=1}^{N}\frac{Y_k}{g^2} = \sum_{k=1}^{N} Z_{\text{in-k}} \\ Z_{\text{load}} &= \sum_{k=1}^{N} Z_{\text{k}} \Rightarrow Y_{\text{input}} = \frac{1}{Y_{\text{load}}/g^2} = g^2 Z_{\text{load}} \\ &= \sum_{k=1}^{N} g^2 Z_{\text{k}} = \sum_{k=1}^{N} Y_{\text{in-k}} \end{aligned} \tag{9}$$

CIRCUIT REALIZATIONS FOR GYRATORS

The 2 G_m Cell Realization

The realization of gyrators in electronic form is quite simple; however, as usual, different circuit realizations are preferable to others, depending on the application. To begin, consider the simplest generic realization comprised of a pair of transcon-

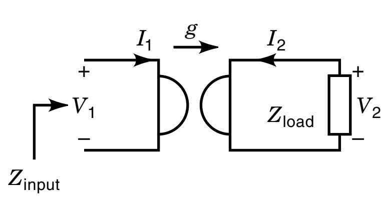

Figure 3. Reflecting load impedance with a gyrator.

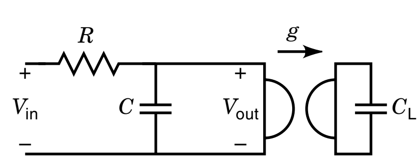

Figure 5. Bandpass filter with inductor replaced by gyrator/capacitor.

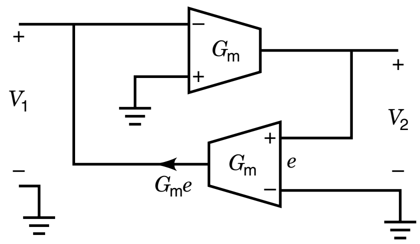

Figure 6. Realization of gyrator using transconductance amplifiers.

ductance amplifiers, as shown in Fig. 6. Each transconductance amplifier is assumed to have infinite input and output impedance, with an output current equal to the transconductance, $G_m = g$, times the input voltage applied to the + and − terminals. The circuit shown in Fig. 6 satisfies the basic two-port relations for a gyrator, given by Eq. (3).

The circuit of Fig. 6 does not implement the most general form of a gyrator, since both ports of the gyrator realization in the figure have ground in common. Therefore, only ground referenced impedances may be transformed as described elsewhere in this article. This limitation stems from the fact that the transconductors in Fig. 6 have single-ended outputs. If differential input/differential output transconductors are used then a general floating gyrator realization is created—that is, a gyrator whose ports need not be referenced in any way to ground.

Unfortunately, the realization of fully differential gyrators is not easy. In general, this realization requires more circuitry and the management of common mode signals. Figure 7 shows how a floating gyrator can be realized using single-ended transconductors; however, this circuit suffers from common mode problems. An analysis of this structure yields the following results:

$$\begin{aligned} I_{2+} = -gV_{1+};\quad I_{2-} - -gV_{1-};\quad I_{1+} = gV_{2+};\quad I_{1-} - gV_{2-} \\ \Rightarrow I_2 = I_{2+} = -I_{2-} = -g(V_{1+} - V_{1-}) = -gV_1 \\ I_1 = I_{1+} = -I_{1-} = g(V_{2+} - V_{2-}) = gV_2 \end{aligned} \tag{10}$$

The crucial assumption embodied in Eq. (10) is that + and − currents are equal and opposite one another. This can only happen if the + and − input voltages at the ports are exactly equal and opposite. Since this special case cannot be relied upon in practice, additional circuitry must be added to deal with any common mode current component. To do this compensation, more transconductors can be added to process the average voltage at each port. As one might expect, this additional circuitry is an unwelcome addition to the design. As a result, this idea is rarely found in practical designs.

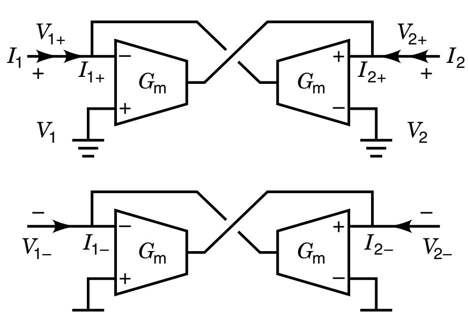

Figure 7. Floating gyrator implementation.

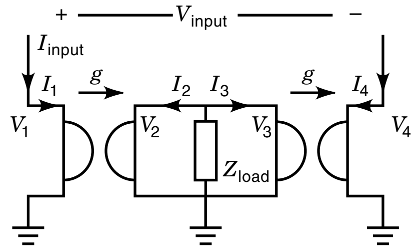

Figure 8. Floating gyrator realization.

There is an alternative for the simulation of a floating inductor using gyrators. A pair of gyrators is used with a grounded impedance, Z_{load}, as shown in Fig. 8. The equations describing this system are given by,

$$\begin{aligned} I_2 &= gV_1;\quad I_1 = -gV_2;\quad I_4 = gV_3;\quad I_3 = -gV_4 \\ V_{\text{load}} &= V_2 = V_3;\quad I_{\text{load}} = -(I_2 + I_3) \\ V_{\text{input}} &= V_1 - V_4;\quad I_{\text{input}} = I_1 = -I_4 \\ Z_{\text{input}} &= \frac{V_{\text{input}}}{I_{\text{input}}} = \frac{V_1 - V_4}{I_1} = \frac{I_2/g - I_3/(-g)}{-gV_2} \\ &= \frac{1}{g^2}\frac{I_2 + I_3}{-V_2} = \frac{1}{g^2}\frac{I_{\text{load}}}{V_{\text{load}}} \\ &= \frac{1}{g^2 Z_{\text{load}}} = \frac{1}{g^2}Y_{\text{load}} \end{aligned} \tag{11}$$

Clearly, if a grounded capacitor is used as the grounded load in Fig. 8, then a floating simulated inductor is realized. The obvious benefit of this realization is that it requires only single-ended transconductance amplifiers, configured as in Fig. 6, and a grounded internal load to obtain a floating input port. Observe that no common mode problems exist with this realization since common mode signals at the input port cause cancelling currents at the grounded port. Even in practice, with unmatched gyrators, there is a negative feedback effect regarding common mode errors which is highly desirable. The network of Fig. 8 is the preferred realization of floating inductors using capacitors and gyrators. Reference 8 represents some original work in this area.

Realization With Operational Amplifiers

Gyrators may be realized with operational amplifiers; however, modifications must be made to account for the fact that these are voltage controlled voltage sources. A voltage controlled current source (VCCS) may be created using an op amp, as is well known, using the circuit of Fig. 9. A gyrator can then be realized with a second VCCS preceded by an inverter, recalling that the gains in different directions have opposite signs. A more clever variation of this idea is shown in Fig. 10, where only two op amps are required to implement the entire gyrator. Of course, this gyrator is ground referenced as, for example, is the one in Fig. 6. Floating gyrator structures can be implemented using the ideas stated, and similarly, a floating inductor may be synthesized via a pair of ground-referenced gyrators implemented with op amps and a

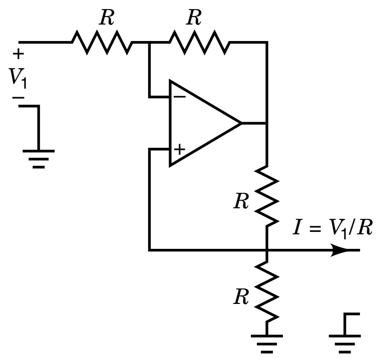

Figure 9. VCCS realization using an op amp.

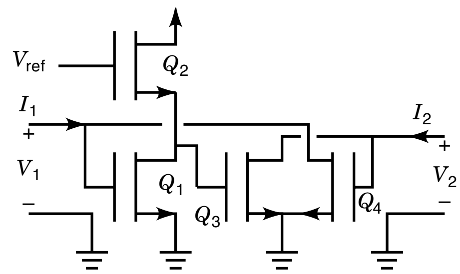

Figure 11. Transistor realization of a gyrator.

grounded capacitor. References 6 and 7 give more discussion of the topics in the last two sections.

Other Realizations

Gyrators may be realized with any active circuitry that can implement either a VCCS or a CCVS (current controlled voltage source). For example, a simple transistor level realization for a gyrator appears in Fig. 11. Q1–Q3 create a first transconductance amplifier, and Q4 implements an inverting transconductance amplifier. The signal levels must be restricted with such an implementation due to the nonlinearity of the transistor junctions.

Alternatively, the transconductance amplifiers comprising the gyrator may be implemented using operational transconductance amplifiers (OTAs). OTAs are essentially bipolar differential pairs loaded with current sources in such a way as to create a nearly ideal transconductance amplifier. An example of an integrated version of an OTA is the LM3080 integrated circuit manufactured by National Semiconductor Corp. of Santa Clara, CA. An important feature of OTAs is that their transconductance can be tuned over a wide range by varying a control current. As a result, gyrators made using OTAs are electronically tunable. This is quite desirable in applications where one wishes to electronically tune a filter. An example of this capability is given later.

Of course, there are limitations imposed by the use of OTAs since these circuits are not ideal in practice. Specifically, they suffer from finite input and output impedance, and these impedances vary as a function of the transconductance. As a result, the tuning range of OTA tuned filters can be limited. Furthermore, at higher frequencies the complexity of OTAs introduces unwanted phase shift which degrades the behavior of the gyrator, as well as compromising the usable tuning range. Finally, these circuits become quite nonlinear for inputs above a few tens of millivolts, which limits the dynamic range of the resulting filters.

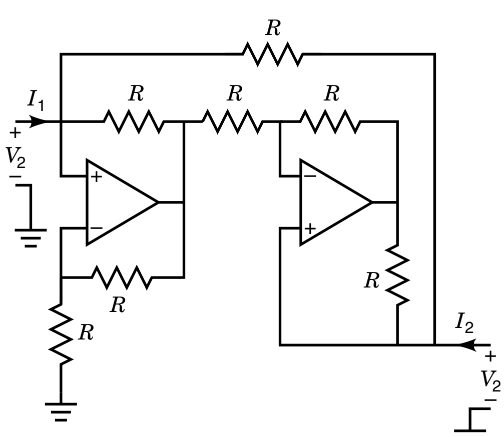

Figure 10. Gyrator realization using op amps.

FILTER REALIZATION USING GYRATORS

This section considers the general problem of filter synthesis based on gyrators. Basically, synthesis with gyrators involves either the substitution of inductors, in the practical case, or the partition of state equations.

Replacement of Inductors in Ladder Networks

The replacement of inductors in filters has already been implied. In this section, the idea is generalized. Consider the case where a prototype passive *RLC* filter has been specified. This is usually done, starting from a filter specification, using filter design tables or software to produce one of a variety of passive filter structures. Special cases will be considered in the discussion that follows. All other cases are obvious variations.

The first case considered is that of a doubly terminated highpass *RLC* ladder network, shown schematically in Fig. 12. It is desirable to replace the grounded inductors with gyrator/capacitor combinations. Given the prototype design values for the inductors, L_k, for $k = 1$ to N, one simply replaces each grounded inductor with one gyrator terminated with a capacitance, C_k, given by the formula,

$$C_k = g_k^2 L_k \tag{12}$$

where g_k represents the gyration constant for the kth gyrator. In practice, all gyration constants might be chosen to be equal

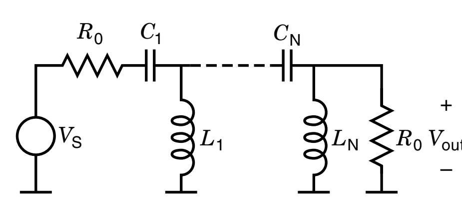

Figure 12. Doubly terminated passive ladder highpass filter.

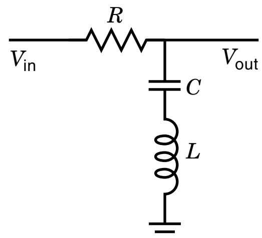

Figure 13. *RLC* notch filter.

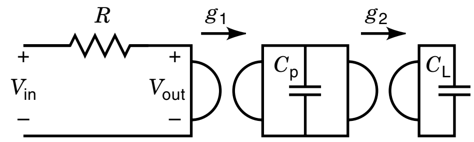

Figure 15. Replacement of series *LC* with two gyrators and two capacitors.

for reasons of simplicity in the circuit design, and possibly for the purpose of optimizing noise and distortion performance. The resulting gyrator-based implementation is now an active filter containing $2N$ capacitors and N gyrators.

The doubly terminated lowpass filter structure obtained by swapping the positions of the inductors and the capacitors in the highpass filter of Fig. 12 is the dual filter network to that previously shown. Again, each of the inductors can be replaced by a gyrator loaded by a capacitor, whose value is computed using Eq. (12). However, this time the gyrator structures will have to be the floating implementation described later. The complexity, in principle, of the final realization will equal that of the highpass example; however, each of the gyrators will require twice as much circuitry for its realization, therefore, the final circuit implementation will require considerably more circuitry as N gets larger.

A more complex filter variation is that of a bandpass *RLC* filter. A doubly terminated bandpass filter can be created starting from the highpass prototype of Fig. 12 by replacing each grounded inductor with a grounded parallel combination of a capacitor and inductor, and each series capacitor with a series combination of a capacitor and an inductor. The resulting filter is of order $4N$, instead of $2N$, as it must be to realize a bandpass equivalent. Again, gyrators may be used to replace each inductor, using methods like Eq. (12). Half of the inductors may be replaced using simple ground referenced gyrators, while the rest must be realized using the floating version.

An interesting final variation of this idea is demonstrated using the notch (bandstop) filter of Fig. 13. Here, the grounded inductor can be replaced by a gyrator and a grounded capacitor as outlined above. An alternative approach is to replace the grounded series *LC* (inductor, capacitor) combination using a gyrator loaded by a grounded parallel *LC* combination as shown in Fig. 14. Then replace the grounded inductor with another gyrator and a grounded capacitor. This is shown in Fig. 15. While this idea may seem to be wasteful in terms of component count, it shows how to exploit the property described elsewhere in this article regarding the conversion of series impedances combinations to parallel impedance combinations. In addition, this circuit has practical value since the pair of gyrators now allows two parameters to be tuned in this active filter realization. Specifically, the filter design equations are given as follows:

$$\frac{V_{\text{out}}}{V_{\in}} = H(s) = \frac{s^2+\omega_0^2}{s^2+(\omega_0/Q)s+\omega_0^2}; \quad \omega_0 = \frac{1}{\sqrt{LC}}; \quad Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$
$$C = g_1^2 L_{\text{p}} = \frac{g_1^2}{g_2^2} C_{\text{L}}; \quad L = \frac{1}{g_1^2} C_{\text{p}} \Rightarrow \omega_0 = \frac{g_2}{\sqrt{C_{\text{L}} C_{\text{p}}}}; \tag{13}$$
$$Q = \frac{g_2}{R g_1^2}\sqrt{\frac{C_{\text{p}}}{C_{\text{L}}}}$$

Using this result, it is easy to see that both the notch frequency, ω_0, and the sharpness of the notch, proportional to Q, can be controlled independently using only the gyration constants, g_1 and g_2. Using an OTA implementation of the gyrators as discussed elsewhere, this tuning is relatively simple.

Synthesis Based on Gyrators

Clearly, the above inductor replacement strategy could be applied in reverse to replace capacitors with inductors; however, this is not a good option in practical cases. This is because capacitors are easier to realize at frequencies below 1 GHz, and capacitors are of generally higher quality than inductors in this range of frequencies. Although inductors may be of greater interest at very high frequencies, the use of gyrators at very high frequencies is limited by the nonideal behavior of the active circuitry used to realize them. It is of interest to note here that, as with any active circuit, there is excess phase shift introduced by the transconductors at high frequencies, making the gyrators take on complex gyration constants at high frequencies, as suggested later in the article. Furthermore, the finite output impedance of the transconductors limits their available dc gain. The net result of these effects is to cause simulated inductors to exhibit a reduced Q at both the low and high end of the frequency spectrum. In some situations this may introduce instability; however, this can be compensated for by careful design.

Another option exists for the generic design of active filters using gyrators. This option can be exercised by casting the equations for a given filter in the G_{m}–C format. This is done by writing the state equations for the desired filter in the standard form,

$$\frac{d}{dt}\boldsymbol{x} = A\boldsymbol{x} + \boldsymbol{b}u; \quad y = \boldsymbol{c}^{\text{T}}\boldsymbol{x} + du; \quad H(s) = \frac{y}{u} = \boldsymbol{c}^{\text{T}}(sI-A)^{-1}\boldsymbol{b} + d$$
$$\text{where } \boldsymbol{x} = (x_1, x_2, \ldots, x_N)^T \tag{14}$$

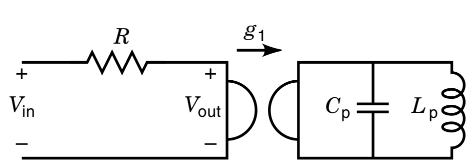

Figure 14. Gyrator based parallel to series conversion.

where the input, u, and the output, y, are assumed scalars, $\boldsymbol{x}$ is the $N \times 1$ state vector, A is the N by N state matrix, $\boldsymbol{b}$ and $\boldsymbol{c}^{\text{T}}$ are N dimensional vectors, and d is a scalar. Now assume that the input, u, is a voltage, and let each of the state variables, x_k, be equated to the voltage, v_k, on some grounded capacitor, C_k. The derivative of this voltage, times the capacitance value, is equal to the current in the respective capacitor. Using this idea the state equations may be converted into current equations of the form,

$$
\begin{gathered}
C\frac{d}{dt}\boldsymbol{v} = \boldsymbol{i}_C = CA\boldsymbol{v} + C\boldsymbol{b}u = G_m\boldsymbol{v} + \boldsymbol{g}_{m0}u \\
\Rightarrow \begin{vmatrix} C_1\ddot{y}_1 \\ C_2\ddot{y}_2 \\ \vdots \\ C_N\ddot{y}_N \end{vmatrix} = \begin{vmatrix} i_{C1} \\ i_{C2} \\ \vdots \\ i_{CN} \end{vmatrix} = \begin{vmatrix} g_{m11} & g_{m12} & \cdots & g_{m1N} \\ g_{m21} & g_{m22} & \cdots & g_{m2N} \\ \vdots & \vdots & \vdots & \vdots \\ g_{mN1} & g_{mN2} & \cdots & g_{mNN} \end{vmatrix} \begin{vmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{vmatrix} + \begin{vmatrix} g_{m01} \\ g_{m02} \\ \vdots \\ g_{m0N} \end{vmatrix} u \\
\text{where } C = \operatorname{diag}(C_1, C_2, \ldots, C_N)
\end{gathered}
\tag{15}
$$

where the dot above a variable denotes time differentiation. The realization of a filter based on these equations produces a filter composed of grounded capacitors, with transconductance amplifiers bridging between the capacitor nodes and the input. The class of G_m–C filters, sometimes referred to as OTA–C filters, is based exclusively on this formulation.

A gyrator-based synthesis is possible by partitioning the G_m matrix into symmetric and skew symmetric matrices. The idea is best described by an example. Suppose a second order version of the g_m–C formulation is given. The G_m matrix can always be decomposed as follows:

$$
\begin{aligned}
G_m &= \begin{vmatrix} g_{m11} & g_{m12} \\ g_{m21} & g_{m22} \end{vmatrix} = G_{m1} + G_{m2} \\
&= \begin{vmatrix} g_{m11} & g_{m12} + g \\ g_{m21} - g & g_{m22} \end{vmatrix} + \begin{vmatrix} 0 & -g \\ g & 0 \end{vmatrix}
\end{aligned}
\tag{16}
$$

where the off-diagonal elements of G_{m1} are equal, making this a symmetric matrix. Clearly, G_{m2} is a skew symmetric matrix. With this partitioning of the transconductance matrix, it is possible to realize the system in Eq. (16) using one reciprocal two-port, characterized by G_{m1}, and a second two-port, characterized by G_{m2}, that is a gyrator. Figure 16 shows the realization associated with this decomposition assuming a single input term in Eq. (15) and a special case for the output—that is, $g_{m02} = 0$ and $y = x_1 = v_1$. The reciprocal two-port can often be realized with only resistors, but in general may require active circuitry.

This special case considered can be further explained with a specific example. Suppose the second order system of Eq. (16) is the bandpass filter described with the following state space description:

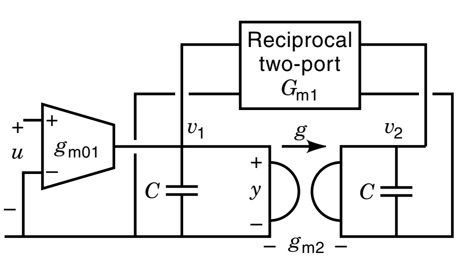

Figure 16. Generic synthesis of a second order G_m-C filter.

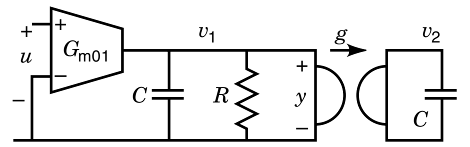

Figure 17. General synthesis realization for a bandpass filter.

$$
\begin{gathered}
\begin{vmatrix} \dot{x}_1 \\ \dot{x}_2 \end{vmatrix} = \begin{vmatrix} -\omega_0/Q & -\omega_0 \\ \omega_0 & 0 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} + \begin{vmatrix} \omega_0 \\ 0 \end{vmatrix} u; \quad y - |1 \quad 0| \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} \\
H(s) = |1 \quad 0| \begin{vmatrix} s + \omega_0/Q & \omega_0 \\ -\omega_0 & s \end{vmatrix}^{-1} \begin{vmatrix} \omega_0 \\ 0 \end{vmatrix} = \frac{s\omega_0}{s^2 + s\omega_0/Q + \omega_0^2}
\end{gathered}
\tag{17}
$$

Let us now assume that the input, u, is a voltage denoted by v_{in}. By equating the state variables, x_1 and x_2, to respective voltages, v_1 and v_2, and scaling each equation by the same capacitance value, C, for convenience, one obtains,

$$
\begin{aligned}
\begin{vmatrix} C\dot{v}_1 \\ C\dot{v}_2 \end{vmatrix} &= \begin{vmatrix} -C\omega_0/Q & -C\omega_0 \\ C\omega_0 & 0 \end{vmatrix} \begin{vmatrix} v_1 \\ v_2 \end{vmatrix} + \begin{vmatrix} C\omega_0 \\ 0 \end{vmatrix} v_{in} \\
&= \begin{vmatrix} -C\omega_0/Q & 0 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} v_1 \\ v_2 \end{vmatrix} + \begin{vmatrix} 0 & -C\omega_0 \\ C\omega_0 & 0 \end{vmatrix} \begin{vmatrix} v_1 \\ v_2 \end{vmatrix} + \begin{vmatrix} C\omega_0 \\ 0 \end{vmatrix} v_{in} \\
&= G_{m1}\boldsymbol{v} + G_{m2}\boldsymbol{v} + \boldsymbol{g}_{m0}v_{in} \\
&\qquad \text{where } \boldsymbol{v} = \begin{vmatrix} v_1 \\ v_2 \end{vmatrix}
\end{aligned}
\tag{18}
$$

Recognizing that $C\omega_0$ has units of conductance, one may easily realize this bandpass filter with only a resistor, implementing G_{m1}, a gyrator, implementing G_{m2}, and a transconductance amplifier realizing the nonzero term in $\boldsymbol{g}_{m0}$. This realization is shown in Fig. 17, where $g_{m01} = C\omega_0$ and $R = Q/C\omega_0$. Observe that this realization is essentially the same as that obtained by replacing the grounded inductor in the bandpass filter of Fig. 4 with a gyrator/capacitor combination.

The generalization of this synthesis technique to arbitrary order systems is straightforward, although cumbersome. In this case, the G_m matrix is again partitioned into symmetric and skew symmetric matrices; however, each related off-diagonal pair of elements in the skew symmetric G_{m2} matrix must be realized with a separate gyrator. This will not be much of a problem if the matrix is sparse, which can often be arranged in setting up the state equations. The N-port gyrator described in, for example, Refs. 9 and 10 can be used to realize the entire G_{m2} matrix at one time.

ADVANCED TOPICS

Energy and Initial Conditions

Gyrators have already been shown to be lossless two-ports. This idea can be extended to show a duality between the energy stored on a capacitor and the energy stored in an inductor. Suppose a capacitor, of value C, is connected to one port of a gyrator. Further suppose that this capacitor is charged to a voltage, V. Then the energy stored in this capacitor is given by $\frac{1}{2}CV^2$. As described earlier, the impedance seen looking into the other port of the gyrator is an equivalent inductor. Given the lossless nature of the gyrator, this equivalent inductor should be expected to have the same apparent stored energy. However, in the case of an inductor the energy stored is $\frac{1}{2}LI^2$, where I is the current flowing in the inductor. A capacitor is in equilibrium with an open circuit across it, and an inductor is in equilibrium with a short across it. Hence, by shorting the port of the gyrator opposite the capacitor, a current flows that will be equal to the equilibrium current in the equivalent inductor. The following analysis shows that the stored energy in the equivalent inductor equals that actually stored on the capacitor.

$$\frac{1}{2}LI^2 = \frac{1}{2}\frac{C}{g^2}(gV)^2 = \frac{1}{2}CV^2 \tag{19}$$

The natural consequence of this energy relationship is that the current at the inductive port cannot change instantaneously, since the voltage at the capacitive port cannot change instantaneously. Hence, initial conditions can be readily translated from one port to another. These facts demonstrate that the gyrator is truly an energy conservative two-port, satisfying any intuition that one might have regarding its operation.

Nonideal Effects

Gyrators, in practice, cannot be made to be ideal. Therefore, practical gyrators are not lossless. Instead they introduce small losses into the system. This is explained by modifying the y-parameter matrix for the gyrator to include diagonal terms. With these terms the two-port is no longer lossless, as is clear from this analysis:

$$\begin{gathered}\boldsymbol{I} = Y\boldsymbol{V} = \begin{vmatrix} g_{11} & -g \\ g & g_{22} \end{vmatrix}\boldsymbol{V} = G\boldsymbol{V} \\ \Rightarrow \boldsymbol{V}^T\boldsymbol{I} = \boldsymbol{V}^T G\boldsymbol{V} = g_{11}V_1^2 + g_{22}V_2^2 \end{gathered} \tag{20}$$

If the diagonal elements, g_{11} and g_{22}, are both positive, then the two-port described in the equation in lossy, since the power delivered to this two-port must be positive. In practice, the loss terms arise naturally from the fact that the transconductors comprising the gyrator are nonideal. For example, the input and output impedance of the transconductors will not be infinite in a practical device. In this case, g_{11} and g_{22} are the nonzero input admittance of the transconductors.

Furthermore, the transfer characteristics will not in general be ideal. As an example, consider the case of the gyrator realized using op amps as in Fig. 10. Suppose the op amps have a finite gain, A. Then the y-parameter matrix can be derived, and is found to be,

$$\begin{gathered} Y = \begin{vmatrix} g_0 & -G \\ G - \delta G & g_0 \end{vmatrix} \\ \text{where } G = \frac{1}{R}; \quad g_0 = 2(1-K)G \\ \delta G = 2(1-K^2)G; \quad K = \frac{1}{1+2/A} \end{gathered} \tag{21}$$

A is the open loop voltage gain of the operational amplifiers. Notice that this Y matrix corresponds to an ideal gyrator when A becomes infinite. Also observe that the finite gain of the op amps has caused the y-parameter matrix to no longer be skew symmetric, which in itself adds loss to the system. Hence, in general, practical gyrators exhibit loss and asymmetry—that is, they lack skew symmetry—in their transfer characteristics.

Another possibility, considering Eq. 20, is for the diagonal elements, g_{11} and g_{22}, to be purely imaginary. In this case, the power computed in Eq. (20) is imaginary, which translates to purely reactive power. When dissipated power is purely reactive, no average power is dissipated. Hence, a gyrator with purely imaginary diagonal elements is still lossless. Such a device could be synthesized by adding reactive elements in series or parallel with the ports of the gyrator, since the diagonal elements, g_{11} and g_{22}, amount to the input admittance looking into the respective ports of the gyrator. Furthermore, stray capacitance or inductance associated with the inputs or the active circuitry making up the gyrator does not contribute to loss.

The Hall Effect Device and Isolators

It has been observed that Hall effect devices implement a lossy gyrator. This is because the physics of these devices is such that the two electric field controlled ports behave as a pair as if they were a gyrator with loss—that is, g_{11} and g_{22} in Eq. (20) are nonzero and not purely imaginary. The physics of such devices is explained in Ref. 11.

Figure 18 shows an interesting usage for a Hall effect gyrator, and in fact any lossy gyrator. In the figure, a gyrator, assumed to have the y-parameter matrix of Eq. (20), has bridging components, R_{P1} and R_{P2}, added around it. Then a pair of sources, V_{S1} and V_{S2}, with respective source resistance, R_{S1} and R_{S2}, are attached as shown. With a little effort the response of this circuit from the sources to the port voltages of the gyrator can be found to be,

$$\begin{gathered} \begin{vmatrix} V_1 \\ V_2 \end{vmatrix} = \alpha \begin{vmatrix} G_{S1}(g_{22}+G_{S2}+G_P) & -G_{S2}(G_P+g) \\ G_{S1}(G_P-g) & G_{S2}(g_{11}+G_{S1}+G_P) \end{vmatrix} \begin{vmatrix} V_1 \\ V_2 \end{vmatrix} \\ \text{where } \alpha = (g_{11}+G_{S1}+G_P)(g_{22}+G_{S2}+G_P)+G_P^2-g^2 \\ G_P = G_{P1}+G_{P2} \end{gathered} \tag{22}$$

By choosing the sum of the bridging elements equal to gyration constant, g, the response at port 2 can be made totally independent of V_{S1}, as opposed to the response at port 1 which will depend upon both sources. This creates a circuit called an isolator which can be found in various applications, especially microwaves and optics.

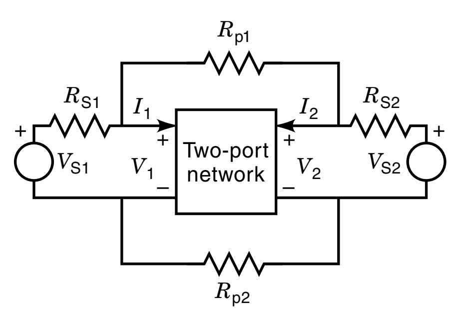

Figure 18. Connection for a lossy gyrator to implement an isolator.

Multiport and Gyrators

The concept of a gyrator need not be restricted solely to two-port networks. In fact, an N-port gyrator can be contrived as a natural extension of the two-port gyrator. As one might expect, the N-port gyrator must inherit the key properties of the two-port type. First, it must be a nonreciprocal lossless network. Secondly, it must reflect impedances in a way similar to the two-port gyrator. In general, N-port gyrators have not found use in electrical systems. For a detailed discussion of such networks refer to Refs. 9 and 10.

There is one special case, however, of an N-port complex gyrator, for $N = 3$, which has found extensive use in microwave systems—namely, the circulator. While practical circulators are quite complex structures, electrically speaking, they can be viewed over a certain range of frequency to be an approximately lossless three-port complex gyrator. Reference 12 describes the three-port y-parameter matrix for a circulator. Specifically,

$$Y = \begin{vmatrix} \alpha & \beta & \gamma \\ -\beta^* & \alpha & \beta \\ -\gamma^* & -\beta^* & \alpha \end{vmatrix} \tag{23}$$

where the superscript * denotes complex conjugation. It is referred to as being complex since the lower triangular matrix part of Y is the negative of the conjugate transpose of the upper triangular part. The power, P, delivered to a three-port having this y-parameter matrix is given by,

$$\begin{aligned} P &= \begin{vmatrix} V_1 & V_2 & V_3 \end{vmatrix} \begin{vmatrix} \alpha & \beta & \gamma \\ -\beta^* & \alpha & \beta \\ -\gamma^* & -\beta^* & \alpha \end{vmatrix} \begin{vmatrix} V_1 \\ V_2 \\ V_3 \end{vmatrix} \\ &= \alpha(V_1^2 + V_2^2 + V_3^2) + (\beta - \beta^*)(V_1V_2 + V_2V_3) + (\gamma - \gamma^*)V_1V_3 \end{aligned} \tag{24}$$

As suggested this power can be made purely reactive if all of the coefficients multiplying the voltage products are purely imaginary. This condition is always met if α is imaginary, since the real parts of β and γ cancel in the final result. Hence, the circulator described by this equation is a lossless three-port given purely imaginary values for α. The circulator is interesting in that the transfer characteristics from port to port when driven by sources—for example, V_{S1}, V_{S2}, and V_{S3}—is similar to the isolator previously described. Specifically, V_{S1} does not affect the port 2 voltage, V_{S2} does not affect the port 3 voltage, and V_{S3} does not affect the port 1 voltage. More details of the design and use of circulators is given in Ref. 13.

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GYRATORS. See MISSILE CONTROL.

GYROMAGNETIC WAVEGUIDES. See FERRITE-LOADED WAVEGUIDES.